APPLIED MATHEMATICS

Part 4:

Fourier Analysis

Wu-ting Tsai

Contents

Chapter 10

Fourier Series, Integrals and Transforms

10.1 Periodic Functions. Trigonometric Series

■ $f(x)$ is periodic if $f(x)$ is defined for $x \in R$ and if $\exists\ p > 0$ such that $f(x+p) = f(x)\ \forall\ x$, p is called period of $f(x)$.

• $f(x+np) = f(x)$ for all integers n
$\Rightarrow$ (np) is also period of $f(x)$.

• If $f(x)$ and $g(x)$ have period p
$\Rightarrow$ $h(x) = af(x) + bg(x)$ also has period p, where a, b are constants.

■ Trigonometric series with period $p = 2\pi$:

$$\begin{aligned} f(x) &= a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \dots \\ &= a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \end{aligned}$$

where a_n and b_n are real constants, called coefficients of the series.

• If $f(x)$ converges
$\Rightarrow$ the sum of the series will be a function with period 2π.

10.2 Fourier Series

Trigonometric series to represent periodic function $f(x)$, where the coefficients are determined from $f(x)$

■ *Euler formulas* for Fourier coefficients a_n and b_n

$$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)}$$

$\boxed{a_0}$:

$$\begin{aligned}
\int_{-\pi}^{\pi} f(x)dx &= \int_{-\pi}^{\pi} \left[a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)\right] dx \\
&= a_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos nx dx + b_n \int_{-\pi}^{\pi} \sin nx dx\right) \\
&= 2\pi a_0 + 0
\end{aligned}$$

$$\Rightarrow \boxed{a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx}$$

$\boxed{a_n\ (n \neq 0)}$: $(n = 1, 2, 3 \ldots)$

$$
\begin{aligned}
& \int_{-\pi}^{\pi} f(x)\cos mx dx \qquad\qquad (m > 0) \\
= & \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\right]\cos mx dx \\
= & \ a_0 \underbrace{\int_{-\pi}^{\pi}\cos m\pi dx}_{0} + \sum_{n=1}^{\infty}\left[a_n \int_{-\pi}^{\pi}\cos nx \cos mx dx + b_n \int_{-\pi}^{\pi}\sin nx \cos mx dx\right] \\
= & \sum_{n=1}^{\infty}\Big[a_n\Big(\frac{1}{2}\underbrace{\int_{-\pi}^{\pi}\cos(n+m)xdx}_{0} + \frac{1}{2}\int_{-\pi}^{\pi}\cos(n-m)xdx\Big) \\
& \quad + b_n\Big(\frac{1}{2}\underbrace{\int_{-\pi}^{\pi}\sin(n+m)xdx}_{0} + \frac{1}{2}\underbrace{\int_{-\pi}^{\pi}\sin(n-m)xdx}_{0}\Big)\Big]
\end{aligned}
$$

$$
\because \frac{1}{2}\int_{-\pi}^{\pi}\cos(n-m)xdx = \begin{cases} \pi & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}
$$

$$
\Rightarrow \int_{-\pi}^{\pi} f(x)\cos mx dx = a_m \pi
$$

$$
\therefore \boxed{a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx dx}
$$

$\boxed{b_n\ (n \neq 0)}$: $(n = 1, 2, 3 \ldots)$

$$
\begin{aligned}
& \int_{-\pi}^{\pi} f(x) \sin mx dx \\
= \ & \int_{-\pi}^{\pi} \left[a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \right] \sin mx dx \\
= \ & a_0 \underbrace{\int_{-\pi}^{\pi} \sin mx dx}_{0} + \sum_{n=1}^{\infty} \Big(a_n \int_{-\pi}^{\pi} \cos nx \sin mx dx + b_n \int_{-\pi}^{\pi} \sin nx \sin mx dx \Big) \\
= \ & \sum_{n=1}^{\infty} \big[a_n \big(\frac{1}{2} \underbrace{\int_{-\pi}^{\pi} \sin(n+m)x dx}_{0} - \frac{1}{2} \underbrace{\int_{-\pi}^{\pi} \sin(n-m)x dx}_{0} \big) \\
& \quad + b_n \big(\frac{1}{2} \int_{-\pi}^{\pi} \cos(n-m)x dx - \frac{1}{2} \underbrace{\int_{-\pi}^{\pi} \cos(n+m)x dx}_{0} \big) \big] \\
= \ & b_m \pi
\end{aligned}
$$

$$\Longrightarrow \boxed{b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx dx} \quad \square$$

<u>*Ex*</u> : Square wave

$$f(x) = \begin{cases} -k & -\pi < x < 0 \\ k & 0 < x < \pi \end{cases} \quad \text{and} \quad f(x+2\pi) = f(x)$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = 0$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx \\ &= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\cos nx dx + \int_{0}^{\pi} k\cos nx dx\right] \\ &= \frac{1}{\pi}\left[-k\left.\frac{\sin nx}{n}\right|_{-\pi}^{0} + k\left.\frac{\sin nx}{n}\right|_{0}^{\pi}\right] = 0 \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx \\ &= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\sin nx dx + \int_{0}^{\pi} k\sin nx dx\right] \\ &= \frac{1}{\pi}\left[k\left.\frac{\cos nx}{n}\right|_{-\pi}^{0} - k\left.\frac{\cos nx}{n}\right|_{0}^{\pi}\right] \\ &= \frac{k}{n\pi}\left[1 - \cos(-n\pi) - \cos n\pi + 1\right] \\ &= \frac{2k}{n\pi}(1 - \cos n\pi) \\ &= \begin{cases} \dfrac{4k}{n\pi} & \text{odd } n \quad (n = 1, 3, 5\ldots) \\ 0 & \text{even } n \quad (n = 2, 4, 6\ldots) \end{cases} \end{aligned}$$

$$\Rightarrow f(x) = \frac{4k}{\pi}(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots) \quad \square$$

■ **Orthogonality** of Trigonometric System:

Functions of $f(x)$ and $g(x)$, $f(x) \neq g(x)$ are orthogonal on some interval $a \leq x \leq b$, if

$$\int_a^b f(x) \cdot g(x) dx = 0$$

For examples, $1, \cos x,\ \sin x,\ \cos 2x,\ \sin 2x,\ \cos 3x,\ \sin 3x, \ldots$ are orthogonal on the interval $-\pi \leq x \leq \pi$, i.e.,

$$\int_{-\pi}^{\pi} \cos mx \cos nx dx = 0 \qquad m \neq n$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx dx = 0 \qquad m \neq n$$

$$\int_{-\pi}^{\pi} \cos mx \sin nx dx = 0$$

- Note that we have used these properties in deriving Euler's formulas. □

■ **Convergence** and Sum of Fourier Series:

$$\text{If} f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \cos nx)$$

⇒ "=" means the series converges and the sum represents $f(x)$. □

Theorem 1 :

(1) periodic function $f(x)$, $f(x) = f(x = 2\pi)$ and

(2) $f(x)$ is "piecewise continuous" on $x \in [-\pi, \pi]$,and

(3) $f(x)$ has left-hand and right-hand derivatives at every $x \in [-\pi, \pi]$

$\Rightarrow$

Fourier series of $f(x)$ converges, and the sum of the series is $f(x)$, except at x_0 where $f(x)$ is discontinuous, at such point the sum is the average of the left-and right-hand limits of $f(x)$ at x_0. □

Note :

If $f(x)$ is discontinuous at $x = x_0$

left-hand limit $$f(x_0 - 0) = \lim_{h \to 0} f(x_0 - h)$$

right-hand limit $$f(x_0 + 0) = \lim_{h \to 0} f(x_0 + h)$$

left-hand derivative $$\left.\frac{df}{dx}\right|_{x_0 - 0} = \lim_{h \to 0} \frac{f(x_0 - h) - f(x_0 - 0)}{-h}$$

right-hand derivative $$\left.\frac{df}{dx}\right|_{x_0 + 0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0 + 0)}{h}$$ □

Ex :

$$f(x) = \begin{cases} x^2 & x < 1 \\ x/2 & x > 1 \end{cases}$$

$$\Rightarrow \quad f(1-0) = 1 \qquad \left.\frac{df}{dx}\right|_{1-0} = 2$$

$$f(1+0) = \frac{1}{2} \qquad \left.\frac{df}{dx}\right|_{1+0} = \frac{1}{2} \quad \square$$

<u>*Ex*</u> : Square wave

$$f(x) = \frac{4k}{\pi}(\sin x + \frac{1}{3}\sin 3x + \ldots)$$

At $x = \frac{\pi}{2}$

$$f(x) = \frac{4k}{\pi}\underbrace{(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots)}_{\pi/4} = k$$

At $x = \frac{3\pi}{2}$

$$f(x) = \frac{4k}{\pi}\underbrace{(-1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \ldots)}_{-\pi/4} = -k$$

At $x = 0$

$$f(x) = \frac{4k}{\pi}(0 + 0 + 0 + \ldots) = 0 \quad \square$$

Proof :

We consider only the convergence of the theorem for a "continuous" function $f(x)$ having continuous 1st and 2nd derivatives.

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \qquad (n = 1, 2, \ldots)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \qquad (n = 1, 2, \ldots)$$

By using integration by part $\int u dv = uv - \int v du$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \\ &= \frac{1}{n\pi} f(x) \underbrace{\sin nx|_{-\pi}^{\pi}}_{0} - \frac{1}{n\pi} \int_{-\pi}^{\pi} f'(x) \sin nx dx \\ &= \frac{1}{n^2\pi} \underbrace{f'(x) \cos nx|_{-\pi}^{\pi}}_{0} - \frac{1}{n^2\pi} \int_{-\pi}^{\pi} f''(x) \cos nx dx \end{aligned}$$

Since $f(x)$ is continuous for $x \in [-\pi, \pi] \Rightarrow |f''(x)| < M$

$$\begin{aligned}
|a_n| &= \frac{1}{n^2\pi}\left|\int_{-\pi}^{\pi} f''(x)\cos nx dx\right| \\
&\leq \frac{1}{n^2\pi}\int_{-\pi}^{\pi} \underbrace{|f''(x)|}_{<M} \cdot \underbrace{|\cos nx|}_{\leq 1} dx \\
&< \frac{1}{n^2\pi}\int_{-\pi}^{\pi} M dx \\
&= \frac{2M}{n^2}
\end{aligned}$$

Similarly

$$|b_n| < \frac{2M}{n^2}$$

$$\begin{aligned}
f(x) &\leq |f(x)| \\
&= \left|a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx)\right| \\
&\leq |a_0| + \sum_{n=1}^{\infty}(|a_n| + |b_n|) \\
&= |a_0| + 2\sum_{n=1}^{\infty}\frac{2M}{n^2} \\
&= \text{finite value}
\end{aligned}$$

$\Rightarrow$ the series converges $\square$

10.3 Functions of Any Period $p = 2L$

So far we only consider periodic functions with period $p = 2\pi$.
For periodic function $f(x)$ with period $p = 2L$, the Fourier series for $f(x)$ is:

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right)$$

where the coefficients are:

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx \quad \square$$

Proof :

Rescaling the variable by $v \equiv \dfrac{\pi x}{L} \Rightarrow x = \dfrac{Lv}{\pi}$

$$x = \pm L \Rightarrow v = \frac{\pi(\pm L)}{L} = \pm\pi$$

$$\begin{aligned} f(x) &= g(v) = g\left(\frac{\pi x}{L}\right) \\ &= a_0 + \sum_{n=1}^{\infty}(a_n \cos nv + b_n \sin nv) \end{aligned}$$

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(v)dv = \frac{1}{2\pi}\int_{-L}^{L} g\left(\frac{\pi x}{L}\right)(\frac{\pi}{L}dx) \\ &= \frac{1}{2L}\int_{-L}^{l} f(x)dx \end{aligned}$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(v)\cos nv dv = \frac{1}{\pi}\int_{-L}^{L} g\left(\frac{\pi x}{L}\right)\cos n\frac{\pi x}{L}\left(\frac{\pi}{L}dx\right) \\ &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx \end{aligned}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(v)\ldots \quad \square$$

<u>Note</u> : Shift of the period

$$\int_{-\pi}^{\pi} \longrightarrow \int_{-\pi+l}^{\pi+l} \qquad \text{eg} : \int_0^{2\pi}$$

$$\int_{-L}^{L} \longrightarrow \int_{-L+l}^{L+l} \qquad \text{eg} : \int_0^{2L}$$

<u>*Ex*</u> : Half-Wave rectifier → clips negative portion of the waves

$$L = \frac{\pi}{\omega} \qquad P = \frac{2\pi}{\omega}$$

$$u(t) = \begin{cases} 0 & -L < t < 0 \\ E \sin \omega t & 0 < t < L \end{cases}$$

$$a_0 = \frac{1}{2L} \int_0^{2L} u(t) dt = \frac{\omega}{2\pi} \int_0^{\pi/\omega} E \sin \omega t dt = \frac{E}{\pi}$$

$$a_n = \frac{1}{L} \int_0^{2L} u(t) \cos \frac{n\pi t}{L} dt = \frac{\omega}{\pi} \int_0^{\pi/\omega} E \sin \omega t \cos n\omega t dt$$

$$= \begin{cases} 0 & n = 1, 3, 5, \ldots \\ \dfrac{-2E}{(n-1)(n+1)\pi} & n = 2, 4, 6, \ldots \end{cases}$$

$$b_n = \frac{1}{L} \int_0^{2L} u(t) \sin \frac{n\pi t}{L} dt = w\pi \int_0^{\pi/w} E \sin wt \sin nwt dt$$

$$= \begin{cases} \dfrac{E}{2} & n = 1 \\ \\ 0 & \text{otherwise} \end{cases} \qquad \square$$

10.4 Even and Odd Functions

- even function: $g(-x) = g(x) \Rightarrow \int_{-L}^{L} g(x)dx = 2\int_0^L g(x)dx$
- odd function: $h(-x) = -h(x) \Rightarrow \int_{-L}^{L} h(x)dx = 0$

■ Products of odd and even functions:

$$g_1(x) \cdot g_2(x) = g_1(-x) \cdot g_2(-x)$$
$$g(x) \cdot h(x) = g(-x) \cdot (-h(-x)) = -g(-x) \cdot h(-x)$$
$$h_1(x) \cdot h_2(x) = (-h_1(-x)) \cdot (-h_2(x)) = h_1(-x) \cdot h_2(-x) \quad \square$$

■ Fourier coefficients of odd and even functions:

$$f(x) = a_0 = \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx \qquad = 0 \quad \text{for odd } f(x)$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx = 0 \quad \text{for odd } f(x)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx = 0 \quad \text{for even } f(x) \quad \square$$

Theorem :

- *Fourier Cosine Series* (for even periodic function $f(x)$ with $p = 2L$)

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x$$

$$a_0 = \frac{1}{L} \int_0^L f(x) dx$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx$$

- *Fourier Sine Series* (for odd periodic function $f(x)$ with period $p = 2L$)

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x$$

$$b_n = \frac{l}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx \quad \square$$

Theorem :

- The Fourier coefficients of $(f_1 + f_2)$ are the sums of the corresponding Fourier coefficients of f_1 and f_2.

- The Fourier coefficients of cf, where c is a real constant, are c times the corresponding Fourier coefficients of f. $\square$

<u>*Ex*</u> : Saw tooth wave

$$f(x) = \underbrace{x}_{f_1} + \underbrace{\pi}_{f_2} \quad \text{for } -\pi < x < \pi, \quad f(x+2\pi) = f(x)$$

$$f_1 = x = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x \quad \text{(odd function)}$$

$$\Rightarrow b_n = \frac{2}{\pi} \int_0^{\pi} f_1(x) \sin nx dx = \frac{2}{\pi} \int_0^{\pi} x \sin nx dx = -\frac{2}{n} \cos n\pi$$

$$f_2 = \pi = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x \quad \text{(even function)}$$

$$\Rightarrow a_0 = \pi, \quad a_n = 0$$

$$\begin{aligned} \therefore f(x) &= \pi + \sum_{n=1}^{\infty} \left(-\frac{2}{n} \cos n\pi \right) \sin \frac{n\pi}{L} x \\ &= \pi + 2 \left(\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x \ldots \right) \end{aligned} \quad \square$$

10.5 Complex Fourier Series

Euler formula:

$$e^{ix} = \cos x + i \sin x$$
$$e^{-ix} = \cos x - i \sin x$$

$$e^{inx} = \cos nx + i \sin nx$$
$$e^{-inx} = \cos nx - i \sin nx$$

$\Rightarrow$

$$\cos nx = \frac{1}{2}(e^{inx} + e^{-inx})$$
$$\sin nx = \frac{1}{2i}(e^{inx} - e^{-inx})$$

For Fourier series:

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \qquad \underline{(a_n, b_n \in R)}$$

$\Rightarrow$

$$\begin{aligned}
& a_n \cos nx + b_n \sin nx \\
&= \frac{1}{2}a_n(e^{inx} + e^{-inx}) + \frac{1}{2i}b_n(e^{inx} - e^{-inx}) \\
&= \frac{1}{2}\underbrace{(a_n - ib_n)}_{\equiv c_n} e^{inx} + \frac{1}{2}\underbrace{(a_n + ib_n)}_{\equiv d_n} e^{-inx}
\end{aligned}$$

$\Rightarrow$

$$f(x) = c_0 + \sum_{n=1}^{\infty}(c_n e^{inx} + d_n e^{-inx})$$

where

$$c_0 = a_0 \in R$$

$$c_n = \frac{1}{2}(a_n - ib_n)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(\cos nx - i\sin nx)dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$$

$$d_n = \frac{1}{2}(a_n + ib_n)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(\cos nx + i\sin nx)dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}dx$$

$$c_n, d_n \in C$$

Note :

$$c_{-n} = d_n$$

$$\Rightarrow f(x) = c_0 + \sum_{n=1}^{\infty}(c_n e^{inx} + c_{-n}e^{-inx})$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

where

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$$

■ Complex form of Fourier series of real function:

For real function $f(x)$ with period $2L$,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{L}}$$

where

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-i\frac{n\pi x}{L}} dx \quad \square$$

10.6 Half-Range Expansion

In practical applications, $f(x)$ are given on some interval only, eg.

■ $f(x)$ is extended as an *even* periodic function $f_1(x)$ with period $2L$.

$\Rightarrow$ $f_1(x)$ can be represented by a Fourier *cosine* series with period $2L$.

■ $f(x)$ is extended as an *odd* periodic function $f_2(x)$ with period $2L$.

$\Rightarrow$ $f_2(x)$ can be represented by a Fourier *sine* series with period $2L$.

■ $f(x)$ is extended as a periodic function $f_3(x)$ with period L.

$\Rightarrow$ $f_3(x)$ can be represented by a Fourier Series with period L.

Note : $f(x) = f_1(x) = f_2(x) = f_3(x)$ ONLY within $0 < x < L$

<u>*Ex*</u> :

$$f(x) = \begin{cases} \dfrac{2k}{L}x & 0 < x < \dfrac{L}{2} \\ \\ \dfrac{2k}{L}(L-x) & \dfrac{L}{2} < x < L \end{cases}$$

(1) Even extension:

$$f_1(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x$$

$$\begin{aligned} a_0 &= \frac{1}{L}\int_0^L f(x)dx \\ &= \frac{1}{L}\left[\int_0^{\frac{L}{2}}\left(\frac{2k}{L}x\right)dx + \int_{\frac{L}{2}}^{L}\frac{2k}{L}(L-x)dx\right] \qquad (\tilde{x} \equiv L-x) \\ &= \frac{1}{L}\left[\frac{2k}{L}\int_0^{\frac{L}{2}} xdx + \frac{2k}{L}\int_{\frac{L}{2}}^{0}\tilde{x}(-d\tilde{x})\right] \\ &= \frac{k}{2} \end{aligned}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx = \ldots = \frac{4k}{n^2\pi^2}\left(2\cos\frac{n\pi}{2} - \cos n\pi - 1\right)$$

(2) Odd extension:

$$f_2(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} = \frac{8k}{n^2\pi^2} \sin \frac{n\pi}{2}$$

(3) Periodic extension with period $p = L$:

$$f_3(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos \frac{2n\pi}{L} x + b_n \sin \frac{2n\pi}{L}) x$$

$$a_0 = \frac{1}{L} \int_0^L f(x) dx$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{2n\pi}{L} dx$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{2n\pi}{L} x dx \quad \square$$

10.7 Forced Oscillations. *Optional*

10.8 Approximation by Trigonometric Polynomials

Application of Fourier series:
(1) solution of partial differential equation,
(2) approximation.

Given a periodic function $f(x)$ with $p = 2\pi$

$$f(x) \cong a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx) \equiv F(x) \quad \text{———} \quad (1)$$

i.e., $f(x)$ is approximated by a trigonometric polynomial of order N.
$\Rightarrow$ Question : Is (1) the "best" approximation to $f(x)$?

Def : "Best" approximation means minimal error E

Def : Error E e.g. $\max |f(x) - F(x)|$
$\Rightarrow$ We choose "total square error" of f for the approximated function F:

$$E = \int_{-\pi}^{\pi} (f - F)^2 dx$$

Note :

- Max $|f - F|$ measures the goodness according to the maximal difference at the particular point.
- $\int_{-\pi}^{\pi} (f - F)^2 dx$ measures the goodness of agreement on the whole interval $-\pi \leq x \leq \pi$.

Proof: $F(x)$ is the best approximation of $f(x)$

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$F(x) = \alpha_0 + \sum_{n=1}^{N}(\alpha_n \cos nx + \beta_n \sin nx)$$

$$\Rightarrow E = \int_{-\pi}^{\pi}(f - F)^2 dx = \int_{-\pi}^{\pi} f^2 dx - 2\int_{-\pi}^{\pi} fF dx + \int_{-\pi}^{\pi} F^2 dx$$

$$\int_{-\pi}^{\pi} fF dx = \pi(2\alpha_0 a_0 + \alpha_1 a_1 + \dots \alpha_N a_N + \beta_1 b_1 + \dots \beta_N b_N)$$

$$\int_{-\pi}^{\pi} F^2 dx = \pi(2\alpha_0^2 + \alpha_1^2 + \dots + \alpha_N^2 + \beta_1 + \dots \beta_N^2)$$

$\because$

$$\int_{-\pi}^{\pi} \begin{pmatrix} \cos nx \\ \sin nx \end{pmatrix} dx = 0$$

$$\int_{-\pi}^{\pi} \begin{pmatrix} \cos mx \cos nx \\ \sin mx \cos nx \end{pmatrix} dx = 0 \qquad m \neq n$$

$$\int_{-\pi}^{\pi} \cos mx \sin nx dx = 0$$

$$\int_{-\pi}^{\pi} \begin{pmatrix} \cos^2 nx \\ \sin^2 nx \end{pmatrix} dx = \pi$$

$$\Rightarrow E = \int_{-\pi}^{\pi} f^2 dx - 2\pi\Big[2\alpha a_0 + \sum_{n=1}^{N}(\alpha_n a_n + \beta_n b_n)\Big] + \pi\Big[2\alpha_0^2 + \sum_{n=1}^{N}(\alpha_n^2 + \beta_n^2)\Big]$$

If we take $\alpha_n = a_n$ and $\beta_n = b_n$

$$\Rightarrow E^* = \int_{-\pi}^{\pi} f^2 dx - \pi\Big[2a^2 + \sum_{n=1}^{N}(a_n^2 + b_n^2)\Big]$$

$$\Rightarrow E - E^* = \pi\Big\{2\underbrace{(\alpha_0 - a_0)^2}_{\geq 0} + \sum_{n=1}^{N}\Big[\underbrace{(\alpha_n - a_n)^2}_{\geq 0} + \underbrace{(\beta_n - \beta_n)^2}_{\geq 0}\Big]\Big\} \geq 0$$

$\Rightarrow E \geq E^*$,

i.e. the error is minimum ($E = E^*$) if and only if $\alpha_0 = a_0, \ \ldots, \ \beta_n = b_n$.

□

Theorem : Minimum square error

The total square error of trigonometric polynomial relative to f on $-\pi \leq x \leq \pi$ is minimum if and only if the coefficients of the trigonometric polynomial are the Fourier coefficients of f.

Notes :

(1) Convergence of the trigonometric polynomial:

$$E^* = \int_{-\pi}^{\pi} f^2 dx - \pi\underbrace{\left[2a^2 + \sum_{n=1}^{N}(a_n^2 + b_n^2)\right]}_{\geq 0}$$

$$\therefore N \nearrow \quad \Rightarrow \quad E^* \searrow$$

$\Rightarrow$ the more terms in partial sum of the Fourier series the better the approximation. □

(2) Bessel inequality:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2 dx \geq 2a_0^2 + \sum_{n=1}^{N}(a_n^2 + b_n^2) \quad \square$$

(3) Parseval's theorem (identity):

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2 dx = 2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \quad \square$$

10.9 Fourier Integrals

- Fourier series $\rightarrow$ periodic function
- Fourier Integral $\rightarrow$ non-periodic function $\quad (p = 2L \rightarrow \infty)$

$$f_L(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos w_n x + b_n \sin w_n x) \qquad w_n = \frac{n\pi}{L}$$

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f_L(\tilde{x}) d\tilde{x}$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f_L(\tilde{x}) \cos w_n \tilde{x} d\tilde{x} \qquad n = 1, 2, \ldots$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f_L(\tilde{x}) \sin w_n \tilde{x} d\tilde{x} \qquad n = 1, 2, \ldots$$

$$\Rightarrow f_L(x) = \frac{1}{2L} \int_{-L}^{L} f_L(\tilde{x}) d\tilde{x}$$
$$+\frac{1}{L} \sum_{n=1}^{\infty} \left[\cos w_n x \int_{-L}^{L} f_L(\tilde{x}) \cos w_n \tilde{x} d\tilde{x} + \sin w_n x \int_{-L}^{L} f_L(\tilde{x}) \sin w_n \tilde{x} d\tilde{x} \right]$$

$$\Delta w \equiv w_{n+1} - w_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L} \quad \Rightarrow \quad \frac{1}{L} = \frac{\triangle w}{\pi}$$

$$\Rightarrow f_L(x) = \frac{1}{2L} \int_{-L}^{L} f_L(\tilde{x}) d\tilde{x}$$
$$+\frac{1}{\pi} \sum_{n=1}^{\infty} \left[(\cos w_n x) \Delta w \int_{-L}^{L} f_L(\tilde{x}) \cos w_n \tilde{x} d\tilde{x} \right.$$
$$\left. + (\sin w_n x) \Delta w \int_{-L}^{L} f_L(\tilde{x}) \sin w_n \tilde{x} d\tilde{x} \right]$$

$$f(x) = \lim_{L\to\infty} f_L(x)$$

$$\lim_{L\to\infty} w_n = \lim_{L\to\infty} \frac{n\pi}{L} \equiv w$$

$$\lim_{L\to\infty} \Delta w = \lim_{L\to\infty} \frac{\pi}{L} \equiv dw$$

$$\lim_{L\to\infty} f_L(x) = \lim_{L\to\infty} \frac{1}{2L}\int_{-L}^{L} f_L(\tilde{x})d\tilde{x}$$
$$+ \lim_{L\to\infty} \frac{1}{\pi}\sum_{n=1}^{\infty} \Delta w \left[(\cos w_n x)\int_{-L}^{L} f_L(\tilde{x})\cos w_n\tilde{x}d\tilde{x}\right.$$
$$\left. + (\sin w_n x)\int_{-L}^{L} f_L(\tilde{x})\sin w_n\tilde{x}d\tilde{x}\right]$$

$$\Rightarrow f(x) = \frac{1}{\pi}\int_0^{\infty} dw\Big[\cos wx \underbrace{\int_{-\infty}^{\infty} f(\tilde{x})\cos w\tilde{x}d\tilde{x}}_{\equiv \pi A(w)} + \sin wx \underbrace{\int_{-\infty}^{\infty} f(\tilde{x})\sin w\tilde{x}d\tilde{x}}_{\equiv \pi B(w)}\Big]$$

$$\therefore f(x) = \int_0^{\infty}[A(w)\cos wx + B(w)\sin wx]dw$$

where

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\tilde{x})\cos w\tilde{x}d\tilde{x}$$

$$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\tilde{x})\sin w\tilde{x}d\tilde{x}$$

This is a representation of $f(x)$ by a Fourier integral, where $f(x)$ is assumed to be *absolutely integrable*, i.e., $\int_{-\infty}^{\infty} |f(x)|\, dx$ exists. $\square$

Theorems : Fourier Integral

If $f(x)$ is

(1) piecewise continuous in every finite interval,

(2) has a right-hand and a left-hand derivative at every point,

(3) absolutely integrable,

then

$f(x)$ can be represented by a Fourier integral.

At the points where $f(x)$ is discontinuous, the value of the Fourier integral equals to the average of the left-hand and right-hand limits of $f(x)$. □

<u>*Ex*</u> :

$$f(x) = \begin{cases} 1 & \text{if} \quad |x| < 1 \\ 0 & \text{if} \quad |x| > 1 \end{cases}$$

$$\begin{aligned} A(w) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(\tilde{x})\cos w\tilde{x}d\tilde{x} \\ &= \frac{1}{\pi}2\int_{0}^{\infty} \cos w\tilde{x}d\tilde{x} \\ &= \frac{2\sin w}{w\pi} \\ B(w) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(\tilde{x})\sin w\tilde{x}d\tilde{x} \\ &= 0 \\ \Rightarrow f(x) &= \int_{0}^{\infty}[A(w)\cos wx + B(w)\sin wx]dw \\ &= \frac{2}{\pi}\int_{0}^{\infty}\frac{\cos wx \sin w}{w}dw \quad \square \end{aligned}$$

$$f(x) = \int_0^\infty [A(w)\cos wx + B(w)\sin wx]dw$$

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\tilde{x})\cos w\tilde{x}d\tilde{x}$$

$$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\tilde{x})\sin w\tilde{x}d\tilde{x}$$

■ Fourier *cosine* integral:

If $f(x)$ is an *even* function, then

$$f(x) = \int_0^\infty A(w)\cos wxdw$$

$$A(w) = \frac{2}{\pi}\int_0^\infty f(\tilde{x})\cos w\tilde{x}d\tilde{x}$$

$$B(w) = 0$$

■ Fourier *sine* integral:

If $f(x)$ is an *odd* function, then

$$f(x) = \int_0^\infty B(w)\sin wxdw$$

$$A(w) = 0$$

$$B(w) = \frac{2}{\pi}\int_0^\infty f(\tilde{x})\sin w\tilde{x}d\tilde{x}$$

10.10 Fourier Cosine and Sine Transforms

Sometimes, it is easier to solve differential and integral equations in the "transformed" space rather than in the original physical space.

⇒ Laplace transform, Fourier transform, Hankel transform,
Hilbert transform, ...

The Fourier cosine integral of a non-periodic even function $f(x)$:

$$f(x) = \int_0^\infty A(w) \cos wx dw \qquad A(w) = \frac{2}{\pi} \int_0^\infty f(\tilde{x}) \cos w\tilde{x} d\tilde{x}$$

⇒ let $A(w) \equiv \sqrt{\frac{2}{\pi}} \hat{f}_c(w)$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c(w) \cos wx dw \qquad \hat{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos wx dx$$

- $\hat{f}_c(w)$ is called *Fourier cosine transform* of $f(x)$
- $f(x)$ is called *inverse Fourier cosine transform* of $\hat{f}_c(w)$

Sometimes we write:

$$\mathcal{F}_c(f) = \hat{f}_c \quad \text{and} \quad \mathcal{F}_c^{-1}(\hat{f}_c) = f$$

Similarly, the Fourier sine integral of a non-periodic odd function $f(x)$:

$$f(x) = \int_0^\infty B(w) \sin wx dw \qquad B(w) = \frac{2}{\pi} \int_0^\infty f(\tilde{x}) \sin w\tilde{x} d\tilde{x}$$

$\Rightarrow$ let $B(w) \equiv \sqrt{\frac{2}{\pi}} \hat{f}_s(w)$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(w) \sin wx dw \qquad \hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin wx dx$$

- $\hat{f}_s(w)$ is called *Fourier sine transform* of $f(x)$
- $f(x)$ is called *inverse Fourier sine transform* of $\hat{f}_s(w)$

Sometimes we write:

$$\mathcal{F}_s(f) = \hat{f}_s \quad \text{and} \quad \mathcal{F}_s^{-1}(\hat{f}_s) = f \quad \square$$

■ Linearity

$$\mathcal{F}_c(af+bg)=\sqrt{\frac{2}{\pi}}\int_0^\infty[af(x)+bg(x)]\cos wx dx$$

$$=a\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos wx dx+b\sqrt{\frac{2}{\pi}}\int_0^\infty g(x)\cos wx dx$$

$$=a\mathcal{F}_c(f)+b\mathcal{F}_c(g)$$

Similarly, $\mathcal{F}_s(af+bg)=a\mathcal{F}_s(f)+b\mathcal{F}_s(g)$ □

■ Transform of Derivative

$$
\begin{aligned}
\mathcal{F}_c(f'(x)) &= \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \cos wx dx \\
&= \sqrt{\frac{2}{\pi}} \left[f(x) \cos wx \,|_{x=0}^{\infty} + w \int_0^\infty f(x) \sin wx dx \right] \\
&= -\sqrt{\frac{2}{\pi}} f(0) + w \mathcal{F}_s(f(x))
\end{aligned}
$$

(Assume $f(x) \to 0$, as $x \to \infty$)

$$
\begin{aligned}
\mathcal{F}_s(f'(x)) &= \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \sin wx dx \\
&= \sqrt{\frac{2}{\pi}} [f(x) \sin wx]_0^\infty - w \int_0^\infty f(x) \cos wx dx \\
&= -w \mathcal{F}_c(f(x))
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{F}_c(f''(x)) &= -\sqrt{\frac{2}{\pi}} f'(0) + w \mathcal{F}_s(f'(x)) \\
&= -\sqrt{\frac{2}{\pi}} f'(0) - w^2 \mathcal{F}_c(f(x))
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{F}_s(f''(x)) &= -w \mathcal{F}_c(f'(x)) \\
&= \sqrt{\frac{2}{\pi}} w f(0) - w^2 \mathcal{F}_s(f(x))
\end{aligned}
$$

10.11 Fourier Transform

$$f(x) = \int_0^\infty [A(w)\cos wx + B(w)\sin wx]dw$$

where

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\tilde{x})\cos w\tilde{x}d\tilde{x}$$

$$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\tilde{x})\sin w\tilde{x}d\tilde{x}$$

$\Rightarrow$

$$\begin{aligned}
f(x) &= \frac{1}{\pi}\int_0^\infty dw \int_{-\infty}^{\infty} f(\tilde{x})[\cos w\tilde{x}\cos wx + \sin w\tilde{x}\sin wx]d\tilde{x} \\
&= \frac{1}{\pi}\int_0^\infty dw \underbrace{\int_{-\infty}^{\infty} d\tilde{x} f(\tilde{x})\cos(wx - w\tilde{x})}_{\text{even function of } w} \qquad (w \in [0,\infty] \to [-\infty,\infty]) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} dw \int_{-\infty}^{\infty} d\tilde{x} f(\tilde{x})[\cos(wx - w\tilde{x}) + \underbrace{i\sin(wx - w\tilde{x})}_{\text{odd function}}] \\
&= \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} dw \int_{-\infty}^{\infty} d\tilde{x} f(\tilde{x}) e^{iw(x-\tilde{x})}}_{\text{complex Fourier integral}}
\end{aligned}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\tilde{x}) e^{-iw\tilde{x}} d\tilde{x} \right]}_{\equiv \hat{f}(w)} e^{iwx} dw$$

$\Rightarrow$ Fourier transform of $f(x)$:

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx$$

Inverse Fourier transform of $\hat{f}(w)$:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{iwx} dw$$

Again $f(x)$ must be
(1) piecewise continuous on every finite interval
(2) $f(x)$ is absolutely integrable

We also write the Fourier transform pair as:

$$\mathcal{F}(f(x)) = \hat{f}(w)$$

$$\mathcal{F}^{-1}(\hat{f}(w)) = f(x) \quad \square$$

Spectrum

Consider the oscillation a spring-mass system:

$$my'' + ky = 0 \quad \text{———} \; (*),$$

where $y(t)$ is the displacement, m is the mass and k is the spring constant.

$$\int (*) y' dt = \underbrace{\frac{1}{2}my'^2 + \frac{1}{2}ky^2}_{\text{total energy}} = E_0 = \text{constant}$$

The solution $y(t)$ of the ordinary differential equation (*) is:

$$y(t) = a_1 \cos\sqrt{\frac{k}{m}}t + b_1 \sin\sqrt{\frac{k}{m}}t$$

$$= c_1 e^{iw_0 t} + c_{-1} e^{-iw_0 t}$$

where $\omega_0 = \sqrt{\frac{k}{m}}, \quad c_1 = \frac{1}{2}(a_1 - ib_1), \quad c_{-1} = \frac{1}{2}(a_1 + ib_1) = c_1^*$

The total energy can be represented by:

$$E_0 = \frac{1}{2}m(c_1 i\omega_0 e^{i\omega_0 t} - c_{-1} i\omega_0 e^{-i\omega_0 t})^2 + \frac{1}{2}k(c_1 e^{i\omega_0 t} + c_{-1} e^{-i\omega_0 t})^2$$

$$= 2kc_1 c_{-1} = 2k\left|c_1\right|^2$$

i.e. $E_0 \propto \left|c_1\right|^2$

If $y = f(x) = \sum_{n=-\infty}^{\infty} c_n e^{iw_n x}$

$\Rightarrow |c_n|^2 \propto$ energy component of frequency w_n

We have a series of $|c_n|^2$ for different w_n. We call this "discrete spectrum".

Similarly, if $y = f(x) = \dfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{iwx} dw$

$\Rightarrow \left|\hat{f}(w)\right|^2 dw \propto$ energy within $[w, w + dw]$

$\Rightarrow \int_{-\infty}^{\infty} \left|\hat{f}(w)\right|^2 \sim$ total energy

We have a function of $\left|\hat{f}(w)\right|^2$. We called this "continuous spectrum". $\square$

Properties of Fourier Transform

- Linearity:

$\mathcal{F}(af + bg) = a\mathcal{F}(f) + b\mathcal{F}(g)$

a and b are constants.

- Fourier Transform of derivative of $f(x)$:

$$\mathcal{F}(f'(x)) = iwF(f(x)) \qquad (f \to 0 \text{ as } |x| \to \infty) \quad \square$$

Proof :

$$\mathcal{F}(f'(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x) e^{-iwx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[f(x) e^{-iwx} \big|_{-\infty}^{\infty} - (-iw) \int_{-\infty}^{\infty} f(x) e^{-iwx} dx \right]$$

$$= 0 + iw\mathcal{F}(f(x)) \quad \square$$

Convolution

For $f(x)$ and $g(x)$, the convolution of f and g
is defined as:

$$h(x) = (f * g) = \int_{-\infty}^{\infty} f(p)g(x-p)dp$$

$$= \int_{-\infty}^{\infty} f(x-p)g(p)dp$$

Theorem: Convolution Theorem

If f and g are piecewise continuous, bounded and absolutely integrable, then

$$\mathcal{F}(f * g) = \sqrt{2\pi}\mathcal{F}(f)\mathcal{F}(g)$$

Proof:

$$\begin{aligned} \mathcal{F}(f * g) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(p)g(x-p)dp \right] e^{-iwx} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(p)g(x-p)e^{-iwx} dx \right] dp \end{aligned}$$

$$x - p \equiv q \Rightarrow x = p + q$$

$$\begin{aligned} \mathcal{F}(f * g) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(p)g(q)e^{-iw(p+q)} dq dp \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(p)e^{-iwp} dp \cdot \int_{-\infty}^{\infty} g(q)e^{-iwq} dq \\ &= \sqrt{2\pi}\mathcal{F}(f)\mathcal{F}(g) \quad \square \end{aligned}$$

$$\mathcal{F}(f * g) = \sqrt{2\pi}\, \underbrace{\mathcal{F}(f)}_{\hat{f}}\, \underbrace{\mathcal{F}(g)}_{\hat{g}}$$

$$\Rightarrow f * g = \int_{-\infty}^{\infty} \hat{f}(w)\hat{g}(w)e^{iwx}dx$$

10.12 Table of Fourier Transforms