No. 1487

MODEL EXPERIMENTS AND THE FORMS OF EMPIRICAL EQUATIONS

BY E. BUCKINGHAM, WASHINGTON, D. C.

Member of the Society

It would often be very advantageous if the behavior under service conditions of projected structures or machines could be predicted from the results of experiments on small models; even roughly approximate information, if thoroughly reliable within its known limits of accuracy, would often prevent serious failures or enable the designer to avoid the use of exaggerated or badly co-ordinated factors of safety and allowances for uncertainty. Experiments on the resistance of ship models have been used in this way for many years with satisfactory results, and have been of great value to the naval architect. On the other hand, it is a familiar fact that in many instances, small scale models do not act at all like their full-sized originals under conditions which seem at first sight to be similar, and that hasty conclusions from model experiments are very unsafe indeed.

2 It is of interest to enquire into the general principles involved in the use of models, and to find, if possible, conditions which must govern such experiments in order that the model shall be similar to its original and its behavior give definite information about the behavior of the original. Just what is meant by *similar*, depends on the nature of the particular case in hand; but there are general rules which show us how to make a model similar to its original, when this is practicable, and which also show us how and why it is often not practicable to fulfill the required conditions. I shall not dilate on the origin of the rules, which are a simple and immediate consequence of familiar physical principles, but will illustrate their meaning and practical application by a few simple examples. There is nothing essentially new in what I have to

Presented at the Spring Meeting, Buffalo, N. Y., June 1915, of THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS.

present, but the subject seems to be rather unfamiliar to engineers in general and is worth discussing because of its frequent practical utility.

THE GENERAL THEOREM

3 To interpret model experiments we have to know how the behavior of the model or the original depends on size; i.e., we must have an equation which describes the behavior of the machine or structure under the conditions of service, and which contains as variables, the size of the machine and the quantities such as speed, applied forces, viscosity of the surrounding medium, etc., which suffice to specify all the essential circumstances of operation. Such an equation is called a physical equation.

4 Except in the most simple cases, the required equation can be found only by experiment, and to set up a satisfactory empirical equation may require very laborious and expensive experimentation if the problem is at all complicated. But the so-called "principle of dimensional homogeneity" is sometimes of great assistance in the following ways: First, it directs our attention to the things we need to measure and keeps our eyes open to the simplifying approximate assumptions we may have to adopt in our work. Second, it reduces the number of separate quantities that have to be varied and gives hints and suggestions as to the most economical way of getting the desired information. Third, it always gives us some information as to the possibilities, and by showing conclusively that certain empirical equations cannot possibly be generally valid, warns us against trusting them too far outside the range of the experiments from which they were deduced and which they may represent quite satisfactorily. Fourth, it sometimes enables us to put equations in such a form that we can use English and metric measurements indiscriminately without wasting time on conversion. And last, and most important for the present discussion, it often enables us to dispense with complete experimental investigations and shows us how very incomplete sets of experiments may give reliable information in particular cases.

5 The principle of dimensional homogeneity states that all the terms of any correct and complete physical equation must have the same dimensions. By this is meant merely that if the numerical value of any term in the equation depends on the size chosen for one of our fundamental units, all the other terms must depend on it in the same way, so that when the size of this unit is changed, the terms

will all be changed in the same ratio and the equation will remain valid, which it would not do otherwise. The necessity for this may be seen from the consideration that the relation, to be described by the equation, among the sizes of certain physical quantities, is a reality which subsists quite independently of any arbitrary choice of units on our part. Hence if the equation is to be complete and correct, the description of facts which it gives must not change when we arbitrarily change from pounds to kilograms or from inches to miles.

6 By means of this principle it may easily be shown[1] that any equation

$$F\,(Q_1, Q_2, \;.\;.\;.\; Q_n) = 0 \qquad [1]$$

describing a relation among the n different kinds of quantity $Q_1, Q_2, \;.\;.\;.\; Q_n$ is always reducible to the form

$$f\,(\Pi_1\; \Pi_2, \;.\;.\;.\; \Pi_{n-k}) = 0, \qquad [2]$$

in which each of the variables Π represents a dimensionless product of the form

$$\Pi = Q^a{}_1\, Q^b{}_2 \;.\;.\;.\; Q^n{}_n; \qquad [3]$$

k is the number of independent fundamental units needed in specifying the units of the n kinds of quantity; and f is some unknown function to be found by experiment.

7 A dimensionless quantity is one of which the numerical value does not change when the sizes of the fundamental units alter, so long as the relations between the derived and the fundamental units are kept unchanged. The simplest example of such a quantity is the ratio of two quantities of the same kind; the ratio of two lengths, for instance, does not depend on the unit we adopt for measuring lengths. Another simple example is the expression $\frac{Dg}{S^2}$ or $D\,g\,S^{-2}$, in which D is the diameter of a fly-wheel, S its peripheral speed, and g the acceleration of gravity. The numerical value of this product, in any particular case, will be the same whether measured by an American using feet and seconds or by a European using meters and seconds, and either of them might change from seconds to minutes without affecting his numerical result, if he changed his derived units of speed and of acceleration accordingly.

8 If there are n separate kinds of quantity but more than one quantity of each kind,—a number of lengths or a number of forces concerned in the relation to be described by the equation, all

[1]Physical Review, vol. 4, p. 345, October 1914.

the quantities of any one kind may be represented by specifying a single one of that kind and the ratios r', r'' . . . of the others to this one. Equation [1] then takes the form

$$F\,(Q_1, Q_2, \;.\;.\;.\; Q_n, r', r'', \;.\;.\;.) = 0 \qquad [4]$$

and this is always reducible to

$$f\,(\Pi_1, \Pi_2, \;.\;.\;.\; \Pi_{n-k}, r', r'' \;.\;.\;.) = 0 \qquad [5]$$

9 This theorem which, for short, I may call the Π theorem, is a convenient statement, for practical use, of the requirement of dimensional homogeneity. We may proceed at once to illustrate its meaning by application to the familiar problem of the flow of fluids through pipes.

THE FLOW OF LIQUIDS IN SMOOTH PIPES

10 When a liquid flows, at a constant rate, through a smooth straight pipe, the pressure gradient G may be expected to depend on diameter D, speed S, and density ρ and viscosity μ of the liquid. So long as the pipe is full and the liquid sensibly incompressible, we do not see anything else foı G to depend on; and unless we have omitted some essential circumstance, these five quantities must be connected by some sort of relation which may be symbolized by writing

$$F\,(G, D, S, \rho, \mu) = 0 \qquad [6]$$

We shall proceed to apply the Π theorem to this equation and compare the results with observed facts.

11 There are 5 separate kinds of quantity involved in the relation, so that $n = 5$, but the units needed for measuring them can all be derived from $k = 3$ fundamental units, so that $n - k = 2$. Hence, whatever the nature of the relation may be, it must be reducible to the form

$$f\,(\Pi_1, \Pi_2) = 0 \qquad [7]$$

containing not five but only two independent variables. Furthermore, we know that any empirical form for [6] which cannot be expressed in the form [7] cannot be generally correct, no matter how good an approximation it may be over a limited range of experiments.

12 To find the two independent dimensionless products Π_1 and Π_2 we must first know the dimensions of the five kinds of quantity in terms of some three which we agree to regard as fundamental. We are not restricted to any particular three, because we are not now concerned with the question of preserving our units by con-

venient primary standards, but only with the interrelations of the five kinds of unit needed in our measurements. Any three mechanical units will serve our purpose if they are independent, i.e., if no one of them can be derived from the others. We adopt mass m, length l, and time t, because these are most commonly used. If we used any other three, such as force, length, and time, the results would be the same, only a little intermediate algebra would be different. We then have the following definitions and dimensional equations:

$$\left.\begin{array}{lll} \text{Pressure gradient} = \dfrac{\text{force}}{\text{area}} \div \text{length} & & G = m\,l^{-2}\,t^{-2} \\ \text{Diameter} = \text{a length} \ldots\ldots\ldots\ldots\ldots & & D = l \\ \text{Speed} = \text{length} \div \text{time} \ldots\ldots\ldots\ldots & & S = l\,t^{-1} \\ \text{Density} = \text{mass} \div \text{volume} \ldots\ldots\ldots & & \rho = m\,l^{-3} \\ \text{Viscosity} = \dfrac{\text{force}}{\text{area}} \div \text{rate of shear}. \,. & & \mu = m\,l^{-1}\,t^{-1} \end{array}\right\} \qquad [8]$$

13 We select any three of these quantities, D, S, ρ, which are independent and might therefore, if we chose, be themselves used as fundamental units, and we write down the equation

$$\Pi_1 = D^x S^y \rho^z G \qquad . \qquad [9]$$

We have then to determine the exponents x, y, z so that Π_1 shall have no dimensions, i.e., so that its unit shall be independent of the sizes of the arbitrary fundamental units m, l, t. This is very easy. Substituting in [9] from [8] we have for the dimensions of Π_1

$$\begin{aligned} \Pi_1 &= l^x . l^y\, t^{-y} . m^z\, l^{-3z} . m\, l^{-2}\, t^{-2} \\ &= l^{x+y-3z-2} . t^{-y-2} . m^{z+1} \end{aligned} \qquad [10]$$

For Π_1 to be dimensionless, the exponent of l which shows how the unit of Π_1 depends on the size of the length unit, must vanish; and similar conditions hold for time and mass. We therefore have the three equations

$$\left.\begin{array}{r} x+y-3z-2=0 \\ -y-2=0 \\ z+1=0 \end{array}\right\} \text{ whence } \left\{\begin{array}{l} x=1 \\ y=-2 \\ z=-1 \end{array}\right.$$

and by substituting in [9] we have

$$\Pi_1 = \frac{D\,G}{\rho\, S^2} \qquad [11]$$

14 To find Π_2 we start with the equation

$$\Pi_2 = D^a\, S^b\, \rho^c\, \mu$$

and follow a similar procedure for determining the exponents a, b, c so that Π_2 shall be dimensionless. The result is

$$\Pi_2 = \frac{\mu}{D\, S\, \rho} \qquad \textbf{[12]}$$

and by equations [**11**] and [**12**], equation [**7**] finally takes the form

$$f\left(\frac{D\, G}{\rho\, S^2},\ \frac{\mu}{D\, S\, \rho}\right) = 0 \qquad \textbf{[13]}$$

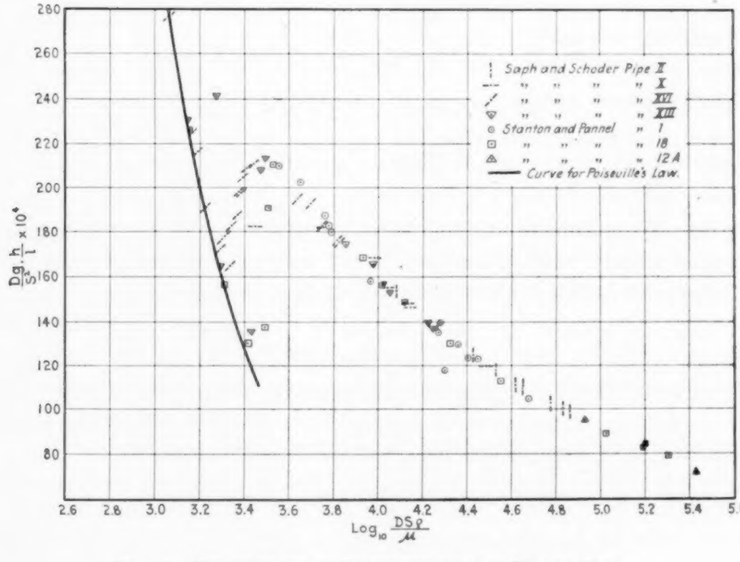

FIG. 1 RESULTS WITH SEAMLESS-DRAWN BRASS PIPE

15 The form of the function f remains to be found from experiment; but whatever it may turn out to be, we know that if there is any relation involving the five quantities of equation [**6**] and no others, it is such that the values of $\frac{D\,G}{\rho S^2}$ and $\frac{\mu}{D\,S\,\rho}$ are connected by a single relation, symbolized by [**13**], and that the size of either of them fixes that of the other,—as may be more conveniently expressed by imagining equation [**13**] to be solved for $\frac{D\,G}{\rho S^2}$ and written in the form

$$\frac{D\,G}{\rho\,S^2} = \varphi\left(\frac{D\,S\,\rho}{\mu}\right) \qquad [14]$$

16 We are now in a position to check results: for if observed values of $\frac{D\,G}{\rho\,S^2}$ are plotted against simultaneous values of $\frac{D\,S\,\rho}{\mu}$, the resulting points should lie along a single curve. If they do not do so, within the experimental errors, we shall know that some essential circumstance must have been overlooked in writing down our original equation [**6**].

17 Fig. 1 shows results obtained with seamless-drawn brass pipe of various diameters. The ordinate is $\frac{D\,G}{\rho\,S^2} \times 10^4$ or its equivalent $\frac{D\,g}{S^2} \cdot \frac{h}{l} \times 10^4$, where $\frac{h}{l}$ is the hydraulic gradient. The abscissa is $\log_{10} \frac{D\,S\,\rho}{\mu}$, which is better than $\frac{D\,S\,\rho}{\mu}$ itself which would make the diagram inconveniently long. Some of the points are from Saph and Schoder's[1] experiments on water. The others are from Stanton and Pannell's[2] experiments on both water and air. Out of the large number of series of experiments, a few were selected at random but so as to cover the whole range of diameters; and of these, only every fourth point was plotted except for $\log_{10} \frac{D\,S\,\rho}{\mu} < 3.5$ when all the points were plotted. If all the hundreds of points had been put in, the diagram would have had to be on a very large scale, but the conclusions to be drawn from it would have been unchanged.

18 It appears first, that the points of all three classes are mixed indiscriminately; and second, that the points are in fact distributed in a well defined band. Hence we conclude that nothing essential was omitted from equation [**6**].

19 The construction of this diagram illustrates one advantage of using dimensionless variables, namely, that it obviates the need of conversions from metric to English units, and vice versa. Saph and Schoder's data are published in English units and in reducing them, the values of $\frac{\mu}{\rho}$ were also expressed in English units. On the other hand, Stanton and Pannell's data are published in c.g.s. units:

[1]Am. Soc. Civ. Engineers; vol. 29, 1903, p. 419.

[2]Phil. Trans. Royal Soc. London: vol. A 214, 1914, p. 199.

but no conversion was needed before plotting, because the numerical value of a dimensionless quantity is independent of the size of the fundamental units so long as we keep the interrelations of the units unchanged, as we do in passing from c.g.s. units to normal English foot, second, pound mass units.

20 Since the object of this section is to illustrate the meaning of the Π theorem in a familiar problem, it would be out of place here to go into an extended discussion of the mathematical form of the function φ and the consequent form of the equation

$$\frac{h}{l}=\frac{S^2}{gD}\varphi\left(\frac{DS\rho}{\mu}\right)$$

and we must also pass over the more interesting subject of rough pipes and the modifications needed in the equations for treating them. We may, however, make three remarks before leaving the present subject:

First. The critical region, where stream line flow changes over to turbulent or hydraulic flow, is shown by the plot to occur at about $\log_{10}\frac{DS\rho}{\mu}=3.3$ to 3.4 or $\frac{DS\rho}{\mu}=2000$ to 2500, as found by Reynolds and others.

Second. If G is proportional to S, as we know experimentally that it is, below the critical speed, the unknown function φ must have the form $\varphi\left(\frac{DS\rho}{\mu}\right)=K\frac{\mu}{DS\rho}$, where K is a constant. Hence equation [**14**] reduces to

$$G=K\frac{S\mu}{D^2}$$

a form of Poiseuille's equation, showing that in this sort of motion the resistance is directly proportional to the viscosity and independent of the density.

Third. If, by reason of high speed, large diameter, or low viscosity, the motion becomes very turbulent, we know from observation that the resistance approaches proportionality to the square of the speed. The function $\varphi\left(\frac{DS\rho}{\mu}\right)$ must then degenerate into a mere constant K_1, and equation [**14**] gives us

$$G=K_1\frac{\rho S^2}{D}$$

the resistance being now sensibly proportional to the density and independent of the viscosity. This agrees perfectly with common sense. The tangential drag between two bodies of liquid moving past each other is merely an effect of cross transmission of momentum between them. When there are no eddies, this interchange of momentum occurs by mixing on a molecular scale, i.e., by diffusion. Hence in stream line motion the resistance depends directly on diffusion and the resulting viscosity. But if the motion is very turbulent, mixing occurs by eddies, and the amount of momentum carried by an eddy of given size depends on the density; the effects of diffusion and its consequence, viscosity, being of very minor importance. We also see why it is that temperature has so little effect on the resistance if the motion is turbulent. Temperature influences viscosity very much and density very little. Hence as turbulence increases and the resistance is more nearly proportional to S^2, the importance of temperature decreases because the importance of viscosity, the only thing sensibly dependent on temperature, decreases.

RESISTANCE OF IMMERSED BODIES AT MODERATE SPEEDS

21 We may next consider the motion of a completely immersed body such as an aeroplane, a dirigible balloon, or a submarine so deeply submerged as to cause no appreciable surface disturbance. Let us enquire how the forces between the fluid and the solid body depend on the various circumstances; and to be specific, let us consider the total head resistance R. The first question is : On what measurable quantities does R depend?

22 To start with, we have the relative speed S of the body and the undisturbed fluid at a distance; we shall suppose S to be constant so that there is no acceleration of the body. Next, we have the size and shape of the body and its orientation with regard to the line of motion: if D is a linear dimension of the body, the shape and orientation may be specified by the ratios r', r'', . . . etc., of a number of other lengths to the particular length D. . . . Finally, we have the mechanical properties of the fluid, its density ρ and viscosity μ. The effects of compressibility do not play any sensible part until the speed approaches that of sound in the medium, and at aeroplane speeds the air behaves very nearly as if incompressible. By moderate speeds we therefore mean speeds which are only a small fraction of the acoustic speed, and for such speeds, compressibility may be left out of account.

23 The condition of total immersion obviates the need to con-

sider either surface tension or the intensity of gravity. In problems of fluid motion, the ratio of viscosity to density appears very often, and it is convenient to represent it by a single symbol $\frac{\mu}{\rho} = \nu$. The quantity ν is known as the *kinematic viscosity* and we may specify the properties of the medium by ρ and ν instead of by ρ and μ as hitherto.

24 If we have not overlooked any important circumstance, there must be some definite quantitative relation connecting the resistance R with the other quantities enumerated, and it may be symbolized by the equation

$$F\,(R, D, S, \rho, \nu, r', r'', \ . \ . \ .) = 0 \qquad [15]$$

which is analogous to equation [6] except that beside the $n = 5$ physical quantities R, D, S, ρ, ν, it contains also a number of dimensionless ratios r.

25 To this equation apply the Π theorem in its general form [5]. The quantities are all mechanical and $k = 3$ fundamental units are required for measuring them. Hence $n - k = 2$ and if such a relation as [15] subsists, it must be of or reducible to the form

$$f\,(\Pi_1, \Pi_2, r', r'', \ . \ . \ .) = 0$$

As in the previous example, the Π's may be found by setting

$$\Pi_1 = D^x S^y \rho^z R; \quad \Pi_2 = D^a S^b \rho^c \nu$$

inserting the known dimensions of D, S, ρ, R, ν; and determining the unknown exponents x, y, z and a, b, c so as to make Π_1 and Π_2 dimensionless. It is unnecessary to give the very simple algebra of the solution: it suffices to note that R is a force and has the dimensions $R = m\,l\,t^{-2}$, while ν has the dimensions $\nu = l^2\,t^{-1}$. The result is

$$\Pi_1 = \frac{R}{\rho\,D^2\,S^2}\,; \quad \Pi_2 = \frac{\nu}{D\,S}$$

and the Π theorem therefore tells us that if such a relation as [15] subsists, it must necessarily be reducible to the form

$$f\left(\frac{R}{\rho\,D^2\,S^2}, \frac{\nu}{D\,S}, r', r'', \ldots\right) = 0 \qquad [16]$$

If this is solved for Π_1 it may be written

$$R = \rho\,D^2\,S^2\,\varphi\left(\frac{D\,S}{\nu}, r', r'', \ldots\right) \qquad [17]$$

in which the nature of the dependence expressed by the unknown function φ remains to be found by experiment if it has to be found at all.

26 In view of the infinite possibilities of varying a body's shape, there are, in the general case, an infinite number of the independent arguments r so that a general determination of the form of φ is impossible. Even if we restrict ourselves to comparatively simple shapes, there will usually be so many r's that the determination of the form of φ would be very laborious. We therefore cut the knot by limiting ourselves to the consideration of one shape and orientation at a time, in other words to studying a series of geometrically similar bodies, anyone of which may be regarded as an increased or diminished model of any other. The separate bodies now differ only in their size, specified by the value of D. The r's are the same for all, i.e., they are mere constants: hence they may be omitted from our equations, and [**17**] reduces, *for any series of geometrically similar bodies*, to the simpler form

$$R = \rho D^2 S^2 \Psi\left(\frac{DS}{\nu}\right) \qquad [\mathbf{18}]$$

in which the unknown function Ψ has only the one argument $\frac{DS}{\nu}$.

27 We might now proceed to find the form of Ψ for bodies of the given shape by plotting observed values of $\frac{R}{\rho D^2 S^2}$ against simultaneous values of $\frac{DS}{\nu}$, drawing a curve, and representing it by an empirical equation. And it may be noted that while the results of experiments on bodies of any size and in any medium might all be utilized, no such variation of D and ν is at all necessary. For $\frac{DS}{\nu}$ may be given any value we please by varying S alone, so that the required information is obtainable from experiments on a single body of the series in a particular fluid.

28 But let us suppose that we are confronted, say, by the practical problem of finding, as economically and quickly as possible, the head resistance of a dirigible balloon of some new and untried shape but of given size and at some given speed. We must if possible avoid the labor of such a complete investigation as outlined above, and equation [**18**] shows us how this may be accomplished by means of model experiments.

29 The original being of length D_0 we may construct a small-

scale model of length D. Let S_0 be the speed of the original and let $S = S_0 \frac{D_0}{D}$. If the model is run at the speed S and the medium is air, as for the original, we shall then have

$$\frac{DS}{\nu} = \frac{D_0 S_0}{\nu_0}, \qquad [19]$$

so that $\Psi\left(\frac{DS}{\nu}\right)$ may be treated as a mere constant. Hence dividing equation [18] for the original by the same equation for the model and setting $\rho = \rho_0$, we have

$$\frac{R_0}{R} = \left(\frac{D_0 S_0}{D S}\right)^2 = 1 \qquad [20]$$

Speeds which are thus chosen so that the unknown function with which the dimensional reasoning leaves us degenerates into a constant, are called *corresponding speeds;* and at corresponding speeds the original and the model are said to be *dynamically similar.* In the present case corresponding speeds are inversely proportional to the linear dimensions, and at corresponding speeds the resistance of the original is equal to that of the model as shown by equation [20].

30 Upon reflexion we see that the foregoing result is of little or no value. For unless the speed S_0 were rather low, any great reduction of scale would require the corresponding speed S of the model to be impracticably high, perhaps even approaching the acoustic speed, so that our disregard of compressibility would no longer be legitimate. But we have still another possibility, namely that of using another medium and so changing $\frac{\nu}{\nu_0}$. The kinematic viscosity of water at ordinary temperatures is from 1/10 to 1/20 that of air according to the temperatures. If we take 1/15 and run the model not in air but in water, we have $\frac{\nu}{\nu_0} = \frac{1}{15}$ and equation [19] then gives us

$$\frac{S}{S_0} = \frac{1}{15}\frac{D_0}{D},$$

so that a model of given size need be run only 1/15 as fast as in air. If experiments are thus made in water, R_0 may be computed from the observed resistance R of the model by using the equation

$$\frac{R_0}{R} = \frac{\rho_0}{\rho}\left(\frac{D_0 S_0}{D S}\right)^2 = 15^2\,\frac{\rho_0}{\rho}$$

or approximately

$$\frac{R_0}{R} = 0.28$$

31 While the foregoing method is strictly correct for speeds at which compressibility is negligible, the process may be much simplified when no high accuracy is required. For it has been established by experiment that with bodies of ordinary shapes at ordinary speeds, i.e., in cases where eddy resistance rather than skin friction is the important thing, the head resistance is very nearly proportional to the square of the speed. But if $R \propto S^2$, the function Ψ is no longer unknown: equation **[18]** shows that it is a constant, i.e., a mere shape factor for bodies of the given shape; and the influence of viscosity vanishes just as it does in the flow of liquids through pipes when there is so much turbulence that the resistance is proportional to the square of the speed. It then follows that

$$R = K \rho D^2 S^2 \qquad \textbf{[21]}$$

and all speeds of a body and its model are corresponding speeds, no matter what the scale ratio or the nature of the medium. All we need, therefore, is a determination of the shape factor K by an experiment on a model of any size, in any convenient medium, after which R may be found from equation **[21]** for the original. Experiments on very small models of dirigibles run in water have been used for determining the relative values of shape factors for various shapes.

32 It may be noted that although R in the equations has represented head resistance, it might equally well have represented the lift of an aeroplane or an inclined dirigible, or any other particular force on an immersed moving body, since we used in our reasoning only the dimensions of R, and all forces have the same dimensions. An equation of the form **[18]** is obtained in any case or, if $R \propto S^2$, an equation of the form **[21]**, the shape factor K depending, of course, on what sort of force is under discussion.

RESISTANCE TO THE FLIGHT OF PROJECTILES

33 When the speed of an immersed body approaches or exceeds that of sound in the medium, as with modern projectiles, the compressibility may become a determining factor in the phenomenon, the energy lost by a high speed projectile being drained away principally in the head and base waves. Compressibility must therefore be recognized in any equations which are to describe what happens.

34 Compressibility and density, together, determine the acoustic speed in the medium, so that this speed C may be used, with the density ρ, in specifying the properties of the medium, instead of using the compressibility itself. Furthermore, since C is a quantity of the same kind as S, the speed of the body, we need not introduce both C and S into our equations but if S appears, we may represent C by its ratio to S, i.e., by $\frac{C}{S}$ or, if we prefer, by $\frac{S}{C}$. In place of equation [**15**] we now have

$$F\left(R, D, S, \rho, \nu, \frac{S}{C}, r', r'', \ldots\right)=0$$

35 The new variable $\frac{S}{C}$ being a mere ratio like the r's, the application of the Π theorem is precisely the same as before, and instead of equation [**17**] we have

$$R=\rho D^2 S^2 \varphi\left(\frac{DS}{\nu}, \frac{S}{C} r', r'', \ldots\right) \qquad [22]$$

To make any practical advance we must again confine our attention to projectiles of some one shape so as to get rid of the shape variables r, and we thus arrive at the equation

$$R=\rho D^2 S^2 \Psi\left(\frac{DS}{\nu}, \frac{S}{C}\right) \qquad [23]$$

36 We have here an instance in which model experiments in a single medium could not, a priori, be expected to furnish reliable information. In order to insure that a projectile and its model shall be dynamically similar, i.e., that the numerical value of the unknown function be the same for both, we must, until it is shown to be needless, make both $\frac{DS}{\nu}$ and $\frac{S}{C}$ constant. Now for a given medium under fixed conditions, the acoustic speed C is fixed; hence the speed S of the model must be the same as that of the original. But ν is also constant for the medium; and it follows that if S, ν, and $\frac{DS}{\nu}$ are to be constant, D also must be the same for the model as for the original. Therefore no change of scale is permissible and we cannot expect to get reliable information from a model.

37 Even if we change the medium from air to water, the case is not much better. The speed of sound in water is about four times that in air, so that if the original were a 15-in. shell at a muzzle velocity of 2500 ft. per sec., a dynamically similar model would need to have a speed in water of about 10,000 ft. per sec.,—a rather formidable

requirement. Assuming this difficulty to be superable, and using water at a temperature such as to make its kinematic viscosity 1/15 of that of air under service conditions, we might then make the model dynamically similar to the original by reducing its diameter to 0.25 in. for we should then have $\frac{DS}{\nu} = \frac{D_0S_0}{\nu_0}$ as well as $\frac{S}{C} = \frac{S_0}{C_0}$. It is conceivable that if we could attain the required initial speed, experiments conducted in this way might furnish some interesting information, equation [**23**] reducing to [**21**] and the shape factor K being found from experiments on models to a scale of about 1/60.

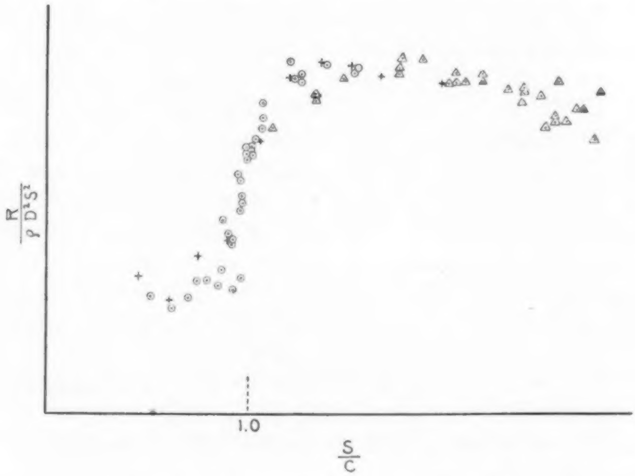

FIG. 2 RESISTANCE OF PROJECTILES

38 While the form of Ψ in [**23**] has not been accurately determined for any shape of projectile, we know that the influence of $\frac{S}{C}$ predominates over that of $\frac{DS}{\nu}$, as might have been expected in view of the turbulence of the motion. For if the observed values of the dimensionless variable $\frac{R}{\rho D^2 S^2}$ for projectiles of the older forms and of various calibers are plotted against values of $\frac{S}{C}$, the resulting points, while rather scattered, do lie more or less along a single curve, which proves that such variations of shape and of $\frac{D S}{\nu}$ as occurred, were of minor importance in comparison with the ratio of

the speed S to the acoustic speed C. Fig. 2 which is reproduced from Cranz's Text Book of Ballistics, exhibits a number of experimental results. The striking thing about the observations is their showing that the development of the wave system on passing the acoustic speed $\frac{S}{C}=1$ increases the resistance about three times, while both above and below this critical region $\frac{R}{\rho D^2 S^2}$ is fairly constant, i.e., the head pressure in air of given density is nearly proportional to the square of the speed.

SCREW PROPELLERS

39 The thrust T of a screw propeller of given shape may be supposed to depend on diameter D, rate of revolution n, and speed of advance S, and on the density ρ and kinematic viscosity ν of the water. If the screw were so deeply immersed that it caused no surface disturbance, these would seem to be all the quantities to be considered; but screws are not usually so deeply immersed as this and we must therefore bring in the weight of the water, i.e., the acceleration of gravity g. We may as well confine our attention, from the start, to propellers of a single shape, immersed to depths proportional to their diameters, and if we have not overlooked any essential factor in the action of the propeller we shall then have

$$F\,(T, D, n, S, \rho, \nu, g)=0 \qquad \textbf{[24]}$$

40 For the measurement of these $n=7$ kinds of quantity, $k=3$ fundamental units are again required, so that $n-k=4$. The Π theorem therefore states that if equation [**24**] is complete, it must be reducible to the form

$$f\,(\Pi_1, \Pi_2, \Pi_3, \Pi_4)=0 \qquad \textbf{[25]}$$

As usual, we select D, S, ρ as convenient independent quantities, and proceed as already illustrated, the only new kinds of quantity having the dimensions $n=t^{-1}$ and $g=l\,t^{-2}$

41 Writing

$$\Pi=D^x S^y \rho^z P,$$

where $P=T$, n, ν, and g successively, and finding the four sets of values which x, y, z must have in order to make Π dimensionless, we obtain the following expressions:

$$\left.\begin{array}{ll} \Pi_1=\dfrac{T}{\rho D^2 S^2}\,; & \Pi_3=\dfrac{\nu}{D\,S} \\[2ex] \Pi_2=\dfrac{D\,n}{S}\,; & \Pi_4=\dfrac{D\,g}{S^2} \end{array}\right\} \qquad \textbf{[26]}$$

Hence if an equation [**24**] subsists, it is certainly of such a nature that it is reducible to the form

$$f\left(\frac{T}{\rho D^2 S^2}, \frac{D n}{S}, \frac{\nu}{D S}, \frac{D g}{S^2}\right)=0 \qquad [27]$$

Being interested more particularly in T, we may solve for Π_1 and write the equation

$$T=\rho\ D^2\ S^2\ \varphi\left(\frac{D n}{S}, \frac{D S}{\nu}, \frac{D g}{S^2}\right), \qquad [28]$$

in which φ is the usual unknown function which always remains after the application of dimensional reasoning, when the number of separate kinds of quantity n is greater than $k+1$.

42 We shall not discuss possibilities of determining the general form of φ, which in this instance has three independent arguments, but shall proceed at once to the question whether experiments on small-scale model propellers can be so carried out and interpreted as to furnish reliable information about full sized propellers of the same shape.

43 To ensure the model being dynamically similar to the original, or in other words, to make certain that φ shall have the same numerical value for both, we must make the three separate arguments of φ have the same values for both, unless it has already been proved that this is unnecessary. The first condition is very simple; for since $D\ n$ is proportional to the tip speed of the blades, $\frac{D n}{S}$ will be constant if the ratio of tip speed to speed of advance is constant. Since the two propellers are of the same shape, this means that corresponding elements of the blades must have the same angle of attack, or that the slip ratios must be the same. The first condition, therefore, for the dynamical similarity of two propellers of the same shape is that they shall be run at the same relative immersion and at the same slip ratio. When this condition is satisfied equation [**28**] reduces to

$$T=\rho D^2 S^2 \Psi\left(\frac{D S}{\nu}, \frac{D g}{S^2}\right) \qquad [29]$$

44 We now encounter a seemingly insuperable obstacle. In practice, we are limited to water so that ν remains nearly constant, and furthermore g is constant. Hence to attain exact

similarity we are directed to keep both $D\,S$ and $\frac{D}{S^2}$ constant, which means that neither D nor S can vary. It follows that no change of size is permissible, and two propellers of different diameters run in the same liquid cannot be made dynamically similar unless it turns out in practice that one of the two arguments of Ψ does not in fact have any sensible influence on the numerical value of Ψ.

45 We therefore cast about to see what may be done in making a justifiable approximation. Viscosity affects the motion of fluids mainly when the motion is quiet; if the motion is very turbulent, viscosity becomes of small or vanishing importance. Now the motion of the water about a screw propeller is excessively turbulent, and if we assume that it is therefore sensibly unaffected by viscosity, we may omit from equation [**29**] the argument $\frac{D\,S}{\nu}$ in which alone the viscosity appears. We thus get a still further simplification to the form

$$T = \rho\, D^2\, S^2\, \Psi_1\left(\frac{D\,g}{S^2}\right) \qquad [\mathbf{30}]$$

46 The rest is easy: if the model of diameter D and original of diameter D_o are run at *corresponding* speeds of advance, S and S_o, such that

$$\frac{D\,g}{S^2} = \frac{D_o\,g}{S_o^2},\ i.e.\ \frac{S}{S_o} = \left(\frac{D}{D_o}\right)^{\frac{1}{2}}, \qquad [\mathbf{31}]$$

the numerical value of Ψ_1 will be the same for both. We then have, by equation [30],

$$\frac{T_o}{T} = \frac{\rho_o}{\rho}\left(\frac{D_o\,S_o}{D\,S}\right)^2;$$

or since $\rho = \rho_o$ we have, utilizing [**31**],

$$\frac{T_o}{T} = \left(\frac{D_o}{D}\right)^3 \qquad [\mathbf{32}]$$

Equation [**32**] states that if geometrically similar propellers are run at the same relative immersion, at the same slip ratio, and at corresponding speeds as defined by [**31**], the thrusts will be proportional to the cubes of the diameters, if the effects of viscosity are unimportant.

47 The validity of this result evidently depends on how far we are justified in the approximation introduced by neglecting viscosity

and so stepping from equation [**29**] to equation [**30**]. It might appear, at first sight, that we were assuming the skin friction of the blades to be unimportant, but the assumption is by no means so violent as this. The only resistance to flow through straight pipes is skin friction, and we know that even for smooth pipes viscosity cuts very little figure in the resistance above the critical speed; while for rougher pipes or more turbulent motion, the influence of viscosity becomes quite negligible. The assumption made above amounts, therefore, only to assuming that if skin friction does play an important role in the operation of a screw propeller, the whole flow of water past the blades is so turbulent that skin friction is sensibly proportional to the square of the speed of the water over the blades. This assumption may not be quite accurate for slow speed propellers but will almost certainly be correct for higher speeds. The justification for the assumption must, of course, rest finally on experiment. So far as the writer knows—his information coming from Rear Admiral D. W. Taylor, U. S. N.—the application of equations [**31**] and [**32**], which amounts to using "Froude's law of comparison," does give correct results, though the available data are not very numerous.

48 For propellers so deeply immersed as to cause no surface disturbance, the weight of the water can have no effect, but only its inertia, so that the value of g can not influence the thrust. It follows that g cannot appear in the equations; $\Psi\left(\frac{Dg}{S^2}\right)$ must be a mere constant shape factor; and [**30**] reduces to

$$T = K\,\rho\,D^2\,S^2 \qquad [\mathbf{33}]$$

Thus we conclude that for propellers of a given shape, running at the same slip ratio and sufficiently deeply immersed, there is no condition of corresponding speeds to be satisfied: the shape factor K may be found by a thrust measurement on the model, and the thrust of the original, running at any speed but with the same slip ratio, may then be computed from equation [**33**] and from the shape factor found for the model at the given slip ratio.

49 Numerous other points might be considered, in addition to the question how the foregoing reasoning, which evidently applies only to propellers advancing into still water, may have to be modified when the propeller is working behind a hull. One such point is cavitation. It is evident without any further algebra that the beginning of cavitation depends on the hydrostatic pressure, among

other things, so that even for deeply immersed propellers, the value of g must occur in the conditions for similarity as regards the cavitation point. When we come to the study of light-weight aero-propellers, we may have to run our propellers so that they are similarly distorted by thrust or by centrifugal force. Or the blade speeds may be so high that the compressibility of the air cannot be disregarded, and then we may have to run propellers of a given shape all at the same tip speed, in order to make them comparable. It is impossible to discuss these questions here but it is also needless because our object is not to discuss the theory of screw propellers but merely to illustrate the use of a method of reasoning.

50 In any case, we have first to ask what are all the physical quantities which may enter into the problem. If we are in doubt about any quantity, it is best to include it in our list and let it drop out later if it proves unimportant. The principle of dimensional homogeneity, in the convenient form of the Π theorem, tells us something about how these various quantities must appear in any correct equation connecting them. Upon examining the result we then see, by inspection, the conditions of dynamical similarity and can judge whether model experiments offer any prospect of advantage or whether they must involve the use of so many doubtful assumptions as to be untrustworthy until based on more complete and general experimental investigations of the phenomena in question.

HEAT TRANSMISSION

51 To avoid leaving one with the erroneous impression that the use of the principle of dimensional homogeneity is confined to the field of mechanics, it will be worth while to treat an illustrative problem which involves thermal quantities. In order not to run to intolerable lengths, we must introduce various simplifying restrictions and use very roughly approximate data as if they were exact; but if these limitations are clearly recognized they will not impair the illustrative value of the treatment.

52 Let us suppose that a homogeneous fluid such as air, water, or superheated steam, is flowing through some metallic apparatus which is hotter or colder than the fluid by an average amount Δ^0. The apparatus and the fluid might, for instance, be a surface condensei and the water in its tubes; or a nest of steam or brine coils and the air passing over them; or simply a straight pipe with hot water running through it. A certain amount of heat H will be transmitted per unit time between the metal and the fluid in ac-

cordance with the temperature head Δ, and we wish to examine the question what this rate of heat transmission depends on and how.

53 The first condition we impose is that the speed of flow S measured, say, at the inlet, shall be great enough that the fluid motion is everywhere very turbulent. We may then regard the motion of the fluid and its convective action as not dependent on viscosity. On the other hand, the speed shall nowhere be so great as to oblige us to introduce compressibility into our equations. We thus exclude from consideration the transmission of heat between a steam-turbine nozzle and its steam jet, but we do not exclude any ordinary practical case. Under the foregoing conditions, the density ρ of the fluid is the only one of its mechanical properties that concerns us.

54 The other properties of the fluid on which H may be supposed to depend are specific heat C, which determines the convective effect of the motion of a given volume, and the thermal conductivity λ, which determines the facility with which heat can pass through the nearly stagnant film which always sticks to the metal surfaces. The greater the speed, in a given apparatus, the more effective will convection be and the more important the specific heat; while simultaneously, the increased scouring action decreases the thickness of the quasi-stagnant film and so decreases the relative importance of the conductivity of the fluid. So long as we have clean metal surfaces we may usually disregard the resistance of the solid parts of the apparatus; but in any event the temperature head Δ now refers to the temperature difference between the fluid and the *surface* of the metal or other solid, either between two specified points or as an average for the whole apparatus, so that the nature and properties of the solid walls do not interest us.

55 We next agree to limit ourselves to consideration of moderate temperature heads such as 200 deg. fahr. or less, within ordinary temperature ranges,—say 0 deg. to 500 deg. fahr. This permits us to introduce several approximations. In the first place, we may disregard variations of the properties of the fluid with temperature, and treat ρ, C, and λ as constants, using average values at the mean temperature. In the second place, we may treat the heat transmission H as dependent only on the difference of temperatures and not directly on the temperatures themselves. And finally, we may without serious error, disregard direct radiation, which might not be legitimate if we were treating of flame in boiler tubes or other instances in which one of the temperatures was very high and the other was not.

56 In spite of these various limitations, it appears upon reflexion that we may still expect our treatment to be approximately correct for the great majority of heat transmitting devices under their ordinary working conditions.

57 The only things not yet mentioned which may be expected to influence H, are shape and size of the apparatus. If we confine our attention to an apparatus of one particular design, the only quantity needed for specifying it is some linear dimension D; and if we have not overlooked anything, there is an equation

$$F\,(H, \Delta, D, S, \rho, C, \lambda) = 0 \qquad [\mathbf{34}]$$

to which we may apply the Π theorem as soon as we know the dimensions of the separate kinds of quantity involved.

58 Three fundamental units are not now sufficient because, in addition to purely mechanical quantities, we now have also thermal quantities to deal with and require a thermal fundamental unit. The most convenient is temperature, which will be denoted by θ and the units for all the $n = 7$ kinds of quantity may be derived from the $k = 4$ fundamental units $[m, l, t, \theta]$. The Π theorem therefore tells us that equation [**34**] must be reducible to the form

$$f\,(\Pi_1, \Pi_2, \Pi_3) = 0 \qquad [\mathbf{35}]$$

59 To determine the forms of the Π's we need the new dimensional equations

$$H = (m\, l^2\, t^{-3})^1 \qquad C = l^2\, t^{-2}\, \theta^{-1},$$
$$\Delta = \theta, \qquad \lambda = m\, l\, t^{-3}\, \theta^{-1},$$

and if we select D, S, ρ, Δ as the quantities for which exponents are to be found, the result of proceeding in the usual manner is to give us, for equation [**35**], the more specific expression

$$f\left(\frac{H}{\rho\, D^2\, S^3}, \frac{C\,\Delta}{S^2}, \frac{\lambda\,\Delta}{D\,\rho\, S^3}\right) = 0,$$

or after solving for H, in which we are particularly interested,

$$H = \rho\, D^2\, S^3\, \varphi\left(\frac{C\,\Delta}{S^2}, \frac{\lambda\,\Delta}{D\,\rho\, S^3}\right) \qquad [\mathbf{36}]$$

60 This is as far as we can go by dimensional reasoning alone, and we must next refer to experimental data, if we can find any suitable, in order to get some information about the form of the unknown function φ. In a new experimental investigation we

[1]A rate of heat flow is merely an amount of power; e.g., 2543 B.t.u. per hour = 1 horse power.

should, of course wish to vary the two quantities $\frac{C\,\Delta}{S^2}$ and $\frac{\lambda\,\Delta}{D\,\rho\,S^3}$ separately. But the quantities S and Δ, which are the easiest to vary in practice, occur in both; while ρ, C and λ, being properties of the fluid, cannot be varied arbitrarily; and changing D requires the use of a larger or smaller model of the apparatus under investigation. What is true of future experiments is true of the past; we must expect to find that the most accurate and most numerous data available refer to the dependence of H on S and Δ.

61 Equation [**36**] is therefore not in proper form for our present purpose, but it may readily be transformed so that either S or Δ shall appear in only one of the arguments of φ. Starting with Δ, we proceed as follows: if

$$\frac{C\,\Delta}{S^2}=\Pi_1 \quad \text{and} \quad \frac{\lambda\,\Delta}{D\,\rho\,S^3}=\Pi_2,$$

we have

$$\Delta=\frac{S^2}{C}\Pi_1 \quad \text{and} \quad \Pi_2=\frac{\lambda}{DS\rho C}\Pi_1,$$

whence we may write

$$\varphi\left(\Pi_1,\,\Pi_2\right)=\varphi\left(\Pi_1,\frac{\lambda}{DS\rho C}\Pi_1\right)$$

But if the value of a certain quantity is determined by the values of two other quantities x and y, it may equally well be specified by stating the values of x and $\frac{y}{x}$; for x and $\frac{y}{x}$ together fix both x and y. To put it in another way, any function $\varphi\ (x,\,y)$ may also be described as some other function $\Psi\ (x,\frac{y}{x})$. Accordingly we may replace the unknown function $\varphi\ (\Pi_1,\,\Pi_2)$ of equation [**36**] by $\Psi\left(\Pi_1,\frac{\lambda}{DS\rho C}\right)$ and so get the equivalent equation

$$H=\rho\,D^2\,S^3\,\Psi\left(\frac{C\,\Delta}{S^2},\frac{\lambda}{DS\rho C}\right) \qquad \textbf{[37]}$$

in which Δ appears in only one argument of the unknown function Ψ.

62 We are now able to refer to experimental facts, and the first we shall use is that when everything else is kept unchanged, the rate of heat transmission in any given apparatus is approximately proportional to the temperature head, when that head is not very

large. Treating this as if it were an exact relation, we conclude that in equation [37], Ψ must contain Δ as a factor and may therefore be written

$$\Psi\left(\frac{C\,\Delta}{S^2},\frac{\lambda}{D\,S\,\rho\,C}\right)=\frac{C\,\Delta}{S^2}\,\Psi_1\left(\frac{\lambda}{D\,S\,\rho\,C}\right),$$

in which the new unknown function Ψ_1 has only a single argument. Equation [37] thus reduces to

$$H=D^2\,S\,\Delta\,\rho\,C\,\Psi_1\left(\frac{\lambda}{D\,S\,\rho\,C}\right) \qquad [38]$$

63 To proceed further by experiment we might change the fluid, thus altering ρ, C, and λ simultaneously; or we might change D by using an apparatus of the same shape but different size; but the most obvious thing to do is to try the effect on H of varying the speed of flow S, while keeping everything else constant. However, even without further experiment we may feel nearly certain that if the flow is very turbulent and the scouring action on the surfaces sufficiently violent, the effect of conductivity λ on H must be small compared with that of specific heat, and that where λ appears in the equation for H it can only be in terms with small exponents. As a rough approximation, we may disregard conductivity altogether and if we do so we have

$$\Psi_1\left(\frac{\lambda}{D\,S\,\rho\,C}\right)=A=\text{constant},$$

so that equation [38] reduces to

$$H=A\,D^2\,S\,\Delta\,\rho\,C, \qquad [39]$$

in which A is a characteristic constant shape factor for apparatus of the given design.

64 To show that the above approximation is not a wild assumption we may refer to the fact that Nusselt,[1] working with compressed gases in round pipes found H nearly proportional to $S^{0.8}$, while Stanton,[2] using liquids, found that H was proportional to a power of S which was a little less than unity,—equation [39] indicating that the exponent should be exactly unity. Nusselt's result would require that [38] should have the form

$$H=A\,D^{1.8}\,S^{0.8}\,\Delta\,\rho^{0.8}\,C^{0.8}\,\lambda^{0.2},$$

in which the effect of conductivity is still perceptible though small,

[1] Zeitschrift des Vereines deutscher Ingenieure *53*, pp. 1750 and 1808; 1909.
[2] Phil. Trans. Royal Soc. London; *A190*, p. 67; 1897.

while that of specific heat is slightly less than on our former simple assumption that the effect of was negligible. Stanton's result would cause still less change in equation [**39**].

65 In the present confused state of the subject of heat transmission we have been obliged, for the sake of completing our illustration, to make use of several admittedly rough approximations. But though equation [**39**], thus obtained, is certainly not exact, it is probably for most ordinary heat transmitting devices nearly enough correct to be worth a little further physical interpretation. If we notice that the whole mass of fluid M, which passes per second, is proportional to the product of cross section, speed, and density, i.e., to $D^2 S \rho$, we have by [**39**]

$$\frac{H}{\Delta} = B M C$$

in which B is a constant shape factor; or in words: the rate of heat transmission per degree temperature head, in an apparatus of given design and with a given fluid, is directly proportional to the mass flow M. Or if the fluid is not always the same, $\frac{H}{\Delta}$ is proportional to the total thermal capacity $M C$ of the fluid which passes through the apparatus in unit time.

66 Another way of putting it is, that since $\frac{H}{MC}$ is the amount by which temperature of the fluid changes while flowing through the apparatus, and since $\frac{H}{MC\Delta} = B =$ constant, the fluid will change its temperature by the same fraction of the temperature difference Δ, regardless of its speed, its nature, and the size of the apparatus, provided only that the shape factor B is constant. This rather startling statement is, at least, not altogether contradicted by experience. For it is known that by forced draught the steaming capacity of a boiler may be greatly increased without much increase in the percentage of heat lost in the flue gases, and this shows that with the increased speed through the flues, the gases have still fallen in temperature by about the same number of degrees, i.e., by about the same fraction of the difference Δ between their mean temperature and that of the tubes, which remains nearly constant.

67 Since all the above relations are only rough approximations, there is no occasion to go farther with them here. But it is hoped that enough has been said to show that the application of dimen-

sional reasoning to the planning of experiments and the interpretation of observed data may sometimes be well worth while in view of the fact that the only labor involved, is at most, the solution of a few simple linear algebraic equations.

CONCLUSION

68 It would be an easy matter to extend this paper indefinitely by treating other problems as, for instance, the static strength of built up structures, the mechanical efficiency of engines or transmission mechanisms, the flow of water in open channels, or the heating of electric generators. The intention has been to select for treatment illustrative examples which had some intrinsic interest and where some tangible result might be obtained. In many cases, the application of the principle of dimensional homogeneity leads to results which are worthless because they merely suggest that we ought to do something which we know is impracticable. But to offset this chance of failure there is the fact that whatever information or suggestions we do get are free, because the application of the theorem is so very simple.

69 The method is not a *theoretical* one in the ordinary sense,—there is nothing hypothetical about it. It is purely algebraic and tells us with certainty, that *if* certain quantities and no others are connected by a physical relation, the equation which describes the relation must be reducible to a certain form: the only chance of mistake is in overlooking some essential factor in the problem. Since the process of reasoning is purely mathematical, we cannot, of course, get out at the end any more than we put in at the beginning when we use our physical common-sense and experience to write down the original list of variables for the problem in hand. But we get out what we put in in a form which often makes it much more available than when it went in.

70 Finally, it may be stated again that the foregoing developments make no claim to essential newness, the purpose having been to call attention to and possibly arouse interest in a very useful kind of reasoning which is by no means so familiarly used as it deserves to be.

APPENDIX

THE DEDUCTION OF THE Π THEOREM

I. Let $Q_1, Q_2, \ldots Q_n$ be a number of physical quantities of different kinds, e.g., a length, a force, a density, etc., which are involved in some physical phenomenon such as the operation of a machine under its working conditions. If these n quantities suffice for describing the phenomenon completely, the value of any one of them is completely determined when the values of the others are given, and this mutual dependence may be stated symbolically by writing the equation

$$F(Q_1, Q_2, \ldots Q_n) = 0 \qquad [1]$$

An equation of this sort, containing symbols which stand for measured numerical values of physical quantities, is a physical equation.

The equations used in engineering are of two kinds. They may be *theoretical*, i.e., based on general principles, like the equation for the energy of a fly wheel or St. Venant's equation for the flow of gases; or they may be *empirical*, i.e., deduced directly from experiments on some particular machine or phenomenon without regard to anything else; formulas for the windage of fly wheels or the loss of head in water mains are empirical equations. But in either case, unless they are mere mathematical formulas such as those of trigonometry, the equations are physical equations and are amenable to the same rules as all other physical equations.

II. Any complete physical equation has the general form

$$\Sigma\, M\, Q^a{}_1\, Q^b{}_2 \ldots Q^n{}_n = 0 \qquad [2]$$

in which the Σ represents summation of a number of terms of the form indicated. The exponents $a, b, \ldots n$ are constants for each term, though they may differ from term to term. There may be any number of terms but the coefficient M of each term is a pure or *dimensionless* number—such as π or $\sqrt{2}$—the value of which does not depend on the sizes of the units used in measuring the Q's, so long as the interrelations of the units among themselves are unchanged.

No purely arithmetical operator such as *log* or *sin* can be applied to an operand which is not a pure number. When such expressions appear to occur in physical equations, as they often do, it is always found on closer examination that the things operated on are in fact only ratios or other dimensionless quantities. The results of the indicated operations are therefore themselves dimensionless numbers independent of the sizes of the units of the Q's, and they may be included in the dimensionless coefficients M.

III. Upon dividing equation [2] through by any one term, we have

$$\Sigma\, N\, Q^{\alpha}{}_1\, Q^{\beta}{}_2 \ldots Q^{\nu}{}_n + 1 = O \qquad [3]$$

Now all the terms of any physical equation must have the same dimensions, and the coefficients N have no dimensions because they are merely ratios of the dimensionless coefficients M. It follows that the exponents of each term of equation [3] must have some such set of values that a dimensional equation of the form

$$[Q^{\alpha}_1 Q^{\beta}_2 \ . \ . \ . \ Q^{\nu}_n] = [1] \qquad [4]$$

is satisfied when the known dimensions of the Q's are inserted.

Let Π_1 represent any dimensionless product of powers of the Q's of the form indicated by (**4**); and let Π_2, Π_3, . . . Π_i be all the other such expressions which can be made up independently by using different sets of values for the exponents. Then the expression ($\Pi_1^{x_1} \Pi_2^{x_2} \ . \ . \ . \ \Pi_i^{x_i}$) is also dimensionless, no matter what the exponents x may be. Consequently equation [**3**] will satisfy the dimensional requirement of having all its terms of the same dimensions—of zero dimensions in this case—if it has the form

$$\Sigma N \Pi_1^{x_1} \Pi_2^{x_2} \ . \ . \ . \ \Pi_i^{x_i} + 1 = 0 \qquad [5]$$

Since the number of terms, the values of the coefficients N, and the values of the exponents x may be anything whatever without affecting the dimensions of any term, the first member of equation [**5**] is merely some entirely indeterminate function of the Π's. Hence the most general and unrestricted form which the physical equation [**1**] can have, subject only to the requirement of dimensional homogeneity, is

$$f(\Pi_1, \Pi_2, \ . \ . \ . \ \Pi_i) = 0 \qquad [6]$$

in which f represents some completely unknown function of which the form remains to be found, either empirically by direct experiment, or theoretically from such general physical principles as may be applicable. In more ordinary language, this means that the statement that a number of physical quantities Q are mutually related as symbolized by equation [**1**], is equivalent to the statement that all the independent dimensionless products Π which can be formed from the Q's, are also mutually related in some definite manner symbolized by equation [**6**].

To illustrate what is meant by *independent* dimensionless products we may consider the two expressions

$$\Pi_1 = \frac{D G}{\rho S^2} = D G \rho^{-1} S^{-2}$$

$$\Pi_2 = \frac{\mu}{D \rho S} = \mu D^{-1} \rho^{-1} S^{-1}$$

which occur in section 3, on the flow of liquids. In the first place, if we raise either of these to any power, we get a new dimensionless quantity; e.g., $\left(\frac{D S \rho}{\mu}\right)^5$ has no dimensions; but it is not *independent* of Π_2 because its numerical value is fixed when that of Π_2 is given. Furthermore, any such combination as

$$\Pi_1 \Pi^{-2}_2 = \frac{D G}{\rho S^2} \times \left(\frac{D S \rho}{\mu}\right)^2 = \frac{D^3 G \rho}{\mu^2}$$

is a new dimensionless product, but it is not independent of Π_1 and Π_2 because its value is fixed by theirs. On the other hand, Π_1 and Π_2 are themselves independent because neither can be obtained from the other.

IV. To measure n kinds of quantity we require n units, but these need not all be adopted arbitrarily for they can in general be derived from, i.e., described or defined in terms of, some smaller number of fundamental units. Let k be the number of fundamental units required for fixing the n kinds of unit needed in measuring the quantities Q_1, Q_2, . . . Q_n. In mechanics all the necessary units can be derived from only three, such as force, length, time, or work, speed, density. Even in the most general problems, dealing with thermal and electromagnetic as well as mechanical quantities k is never greater than five.

If k is the number of fundamental units needed, there is always, among all the n units, at least one set of k which are independent of one another and might themselves be used as fundamental units if we paid no attention to the question of representing and preserving the fundamental units by satisfactory primary standards. Let $[Q_1, Q_2, \ldots Q_k]$ represent such a set and let the remaining $n-k$ units be denoted by $[P_1, P_2, \ldots P_{n-k}.]$ Then each of the P's may be derived from the Q's in accordance with a dimensional equation which may be written

$$[Q^{\alpha}_1 Q^{\beta}_2 \ldots Q^{\kappa}_k . P]=[1] \qquad [7]$$

Since there are $n-k$ of the P's, there are $n-k$ separate independent equations of the form [7], and no more.

If in equation [7] we substitute for P and the Q's their dimensional equivalents in terms of any convenient fundamental units (see equations [9] to [11] of Section 3 in the body of the paper), the requirement that the total exponent of each fundamental unit shall vanish furnishes k independent linear equations which suffice for the determination of the exponents $\alpha, \beta, \ldots \kappa$ of equation [7]. If, after determining these exponents for any particular P, we set

$$\Pi = Q^{\alpha}_1 Q^{\beta}_2 \ldots Q^{\kappa}_k P \qquad [8]$$

the quantity Π satisfies the requirement of being a dimensionless product of the specified form. There are $n-k$ independent equations of the form [7] and therefore the same number of Π's, hence $i=n-k$, and the number of *independent* Π's is k less than the whole number of different kinds of quantity Q.

V. We have hitherto confined our attention to a relation among quantities that are all of different kinds. If several quantities of any one kind are involved in the relation to be described, they may all be specified by the value of any one and the ratios r', r'', etc., of the others to that one. Dimensional considerations cannot tell us anything about the manner in which these dimensionless ratios r appear in the equation which describes the relation, but their possible influence must be indicated, and this may be done in an entirely general way by introducing them as additional independent arguments of the unknown function f. The limitation imposed by the requirement of dimensional homogeneity upon the possible forms of physical equations may therefore be conveniently summarized in the following statement:

Any complete physical equation which describes a relation subsisting among quantities of n different kinds, of which k kinds are independent and not derivable from one another, is reducible to the form

$$f(\Pi_1, \Pi_2, \ldots \Pi_{n-k}, r', r'', \ldots)=0 \qquad [9]$$

in which the r's represent all the independent ratios of quantities of the same kind, and each Π is determinable from a dimensional equation of the form

$$[\Pi]=[Q^{\alpha}_1 Q^{\beta}_2 \ldots Q^{\kappa}_k . P]=[1] \qquad [10]$$

VI. If equation [9] is solved for any one of the Π's, for instance Π_1, it may be written

$$P_1 = Q^{a}_1 Q^{b}_2 \ldots Q^{k}_k \varphi(\Pi_2, \Pi_3, \ldots \Pi_{n-k}, r', r'', \ldots) \qquad [11]$$

in which

$$a=-\alpha_1, \; b=-\beta_1, \text{ etc.}$$

If it is desired to obtain an equation of the form [**11**] with a particular quantity P_1 appearing separately and in the first member only, this quantity should, from the outset, be excluded from the list of quantities which are to be used as the Q's in handling the $n-k$ equations of the form [**10**]. This precaution is not always necessary, but it is always *sufficient* to ensure that the quantity in question shall appear in only a single one of the Π's and shall therefore be separable.

Equation [**9**] may also, of course, be put into the form

$$r' = \Psi\,(\Pi_1, \Pi_2, \;.\;.\;.\; \Pi_{n-k}, r'', r''', \;.\;.\;.)$$

which is sometimes useful.

VII. Since equation [**11**] contains an unknown function φ, the form of which cannot be found by dimensional reasoning, the equation does not give us any definite information in the general case when all the quantities which appear in the second member are allowed to vary independently. But if all the r's are held constant, and if the Q's and $P_2, P_3, \;.\;.\;.\; P_{n-k}$ are allowed to vary, not arbitrarily but only in such ways as will keep the values of $\Pi_2, \Pi_3, \;.\;.\;.\; \Pi_{n-k}$ constant, then we do have a definite statement of the dependence of P_1 on the Q's. For under these conditions, although its general form remains unknown the function φ degenerates into a dimensionless constant N, because all its arguments are constant. Hence under these conditions equation [**11**] assumes the definite form

$$P_1 = N\, Q^{a}{}_1\, Q^{b}{}_2 \;.\;.\;.\; Q^{k}{}_k \qquad [\mathbf{12}]$$

A single experimental measurement of a set of simultaneous values of P_1, and the Q's suffices to determine the numerical value of N; and by equation [**12**] with this experimental value of N, the value of P_1 may be computed, without further experiment, for any other values of the Q's which satisfy the requirement, noted above, of keeping $\Pi_2, \Pi_3, \;.\;.\;.\; \Pi_{n-k}$ and the r's constant.

The chief utility of the principle of dimensional homogeneity is found in its application to problems in which it is practicable to arrange matters so that the r's and Π's of equation [**11**] shall remain constant and the definite equation [**12**] therefore be satisfied. This is what has to be done when we use model experiments for getting information about the behavior of the full sized originals, and the practicability or impracticability of satisfying the required conditions (which is evident upon inspection of the list of Π's and r's) is what determines whether we can or cannot obtain reliable information from models.

DISCUSSION

M. D. Hersey (written). The author has struck the keynote of a new development of technical physics, which will eventually play the same part in mechanical engineering that physical chemistry has begun to play in the chemical industries.

The importance of technical physics, as a branch of subject matter, is already so clearly recognized in Germany that laboratories are being established devoted exclusively to this field. But the development which I think we may now anticipate is something distinct from this, and a natural sequel to it: I refer to the development of technical physics, not as a branch of subject matter, but as a method of reasoning.

It is from such a standpoint that technical physics becomes analogous to physical chemistry which is the planning and interpretation of chemical experiments in the light of thermodynamics and the phase rule; and the Π-theorem is closely analogous to thermodynamics and the phase rule. Thermodynamics affords certain rigid connecting links between seemingly isolated experimental results, while the phase rule tells us the number of degrees of freedom of a chemical system. The Π-theorem likewise affords rigid connecting links which not only serve as a check on the consistency of our results, but may greatly cut down the labor of experimenting. Thus, on applying the Π-theorem to lubrication, we find, under certain conditions, that the coefficient of friction, f, must be some function or other of the *two* variables $\mu n/p$ and D^3n/Q alone; in which μ denotes viscosity, n revolutions per unit time, p bearing pressure, D journal diameter, and Q volume of oil pumped through bearing in unit time. Hence, the same change in f will be produced by a given change in the argument $\mu n/p$, whether this change is, in turn, caused by varying μ in one direction or by varying p in the other direction; and so on. These facts, all implicitly contained in the Π-theorem, can, for the sake of emphasizing our analogy, be expressed by the equations

$$\frac{df}{d\mu/\mu} = -\frac{df}{dp/p} \text{ whence } \frac{df}{dp} = -\frac{\mu}{p}\cdot\frac{df}{d\mu}$$

$$\frac{df}{dD^3/D^3} = -\frac{df}{dQ/Q} \text{ whence } \frac{df}{dD} = -3\frac{Q}{D}\frac{df}{dQ}; \text{ etc.}$$

And, just as the phase rule tells us that the number of degrees of freedom in a chemical system is $F = K - P + 2$, K being the number of components which coexist in P phases, so also the Π-theorem tells us that the number of degrees of freedom in a physical system is $f = p - k - 1$, k being the number of fundamental units needed to describe a relation subsisting among the p physical quantities. For it diminishes by k the number of factors which have to be varied experimentally.

The author has himself stated that the paper contains nothing essentially new. Any illusion to the contrary would be an impediment to the successful use of the methods presented. The kernel of the paper is a theorem which is merely a restatement of the requirements of dimensional homogeneity, announced by Fourier nearly a hundred years ago, and extensively used by Rayleigh and others. But the fact that the paper contains nothing essentially new does not diminish its value. Gibbs' phase rule, too, was new only in form, not in substance, yet it served as the crystallizing influence which caused an immense number of latent ideas to fall into line, and we may expect the Π-theorem to play a similar rôle.

This inevitable development of technical physics into a unified branch of science, which will acquire the same fundamental place in the engineering curriculum that physical chemistry now holds in the chemical curriculum, can be facilitated if writers on the problems of hydro- and aerodynamics, heat transmission and the like will be as introspective as possible, explicitly calling attention not only to their results, but to their methods of reasoning as well. For in every successful artifice of reasoning, there must be some element which is universal and capable of being generalized and made into a working tool.

Melach I. Nusim (written). The conditions for similarity, discussed in the paper, have been noted and applied with great advantage by designers of centrifugal compressors and pumps. Two centrifugal compressors, if they are geometrically similar, have the same efficiency provided the following relation is maintained between the rated flow, Q, the peripheral speed of the impeller, S, and the impeller diameter, D:

$$Q_1/S_1D_1^2 = Q_2/S_2D_2^2 = \text{constant}$$

The experimental data on one particular size of compressor can be utilized to predict with accuracy the performance of a number of sets, provided the compressors are made geometrically similar and rated according to the relation mentioned above. In terms of the mean effective pressure, P, the r.p.m., N, and the flow, Q (volume per unit of time), the relation is equivalent to

$$QN^2/P^{\frac{3}{2}} = \text{constant}$$

A. R. Dodge said he had investigated the drop in pressure of superheated steam under similar conditions to those of Stanton's and Pannel's experiments on air and water illustrated in Fig. 1, using a 1-in. smooth drawn brass pipe, steam jacketed. Over the range covered (between ordinates of 3.8 to 5.6) the results coincided exactly with those shown in this curve, showing that the curve applies for steam, in addition to air, water and oil. It ought, therefore, to apply for any fluid, and simplify existing formulae for pressure drop.

He had also made a number of steam tests with commercial iron pipe 2 to 8 in. in diameter, and found that the curves were parallel to and above that shown in Fig. 1, the distance above depending on the roughness of the pipe.

John R. Freeman said that he possessed data of his own experiments with ordinary rough pipes, which might be of value in solving the problems presented. He said he had conducted a very extended range of experiments, in 1893, on pipes of all degrees of smoothness, from seamless brass pipe to exceedingly rough pipe.

L. W. Wallace said that in connection with investigations to determine certain facts in reference to locomotive sparks, it became necessary to know to what height sparks are ejected from locomotives under various operating conditions, and he asked whether experiments with a specially designed model locomotive would give data that would be comparable with the actual height the sparks would be thrown, the size of the sparks, etc.

The Author. In reply to Mr. Wallace, it is conceivable that model experiments might be so devised as to furnish the desired information, but the difficulties appear, at first sight, rather formidable. It is impossible to say off-hand, before examining the problem carefully, whether an attempt to solve it in this way would have any prospect of success. It would seem much simpler to study the actual emission of sparks from a locomotive by making runs at night and taking photographs—possibly kinematograph records—from two points on the train, one close to the locomotive and one much farther back.

In reply to Mr. Freeman, while Stanton and Pannell's experiments on smooth brass pipes were possibly somewhat more accurate than Saph and Schoder's, the latter had also worked with galvanized pipes. In trying to get an equation which could be made to represent the resistance of both smooth and rough pipes by varying only a single quantity—representing the roughness—the author had therefore used Saph and Schoder's results exclusively, because in a preliminary study consistency was more important than accuracy. He had found, however, that the data were not sufficient for his purpose, and it would be a matter of great interest to him if he were priviledged to examine Mr. Freeman's experimental results.

In reply to Mr. Dodge, it must be a satisfaction to all concerned with the subject of pipe resistance that the results of his wide experience with steam agrees so well with those obtained by Saph and Schoder for water, and by Stanton and Pannell for both water and air. We may safely conclude that the basis of physical ideas from which the dimensional treatment starts is sensibly correct, and that no important element in the problem has been overlooked. The results obtained by dimensional reasoning are so instructive and the problem of pipe resistance so important, that the confirmation which Mr. Dodge has given is a valuable contribution to the subject.

The example, brought forward by Mr. Nusim, of the practical utilization of the notion of dynamical similarity is very interesting. The author's object in presenting his paper was to call attention to the method which, he is convinced, will in time come to be one of the engineer's handy tools, like the two laws of thermodynamics. But since he is aware that his opinion of the value of the method may be received somewhat sceptically by professional engineers, testimony in its favor from one engaged in practical designing work is doubly welcome.

In reply to Mr. Hersey, mechanical engineering is an art, not a science; and the ability and imagination of the individual engineer will always be its most important element. But common sense tells the engineer to get as much outside help as he can in solving his problems, and one source of such help is physics. As Mr. Hersey points out, the aid to be got from physics does not consist merely in new determinations of physical constants or in experimental investigations of physical problems which are of special interest to engineers. It consists also in the systematic use of the scientific method of physics in analyzing problems, planning experiments, and coördinating known facts so as to bring to bear on any new problem all the available knowledge, of whatever sort and wherever obtained, which may seem to be pertinent. This is the technical physics which is destined not only to work in its laboratories on problems presented by engineers, but to be recognized as an inseparable companion of sound and progressive mechanical engineering.