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A Finite Element Solution for a Three Dimensional Inviscid Flow over a Rectangular Model Car

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ABSTRACT

A variational method is employed for analyzing a steady three-dimensional inviscid compressible flow-field starting with the energy formulation. The Clebsch transformation of velocity vectors and a set of governing equations using Lagrangian multipliers and entropy are derived. The resulting set of equations is equivalent to the classic Euler equations in terms of primitive variables. This formulation provides a solution scheme for solving potential and Euler equations. The resulting set of variational equations is approximated by an isoparametric finite element method. The three dimensional inviscid flow over a rectangular model car has been analyzed by the present finite element computer code.

Keywords: variational method, clebsch transformation, three-dimensional

I. Introduction

Development of a variational principle is an important issue in the area of fluid dynamics. It is known that a direct variational formulation for a fluid problem can be derived over the Lagrangian coordinate system. As far as the Eulerian formulation is concerned, additional effort must be made to construct a variational principle.

The mathematic foundation of constructing a variational principle for the inviscid flow problem over the Eulerian coordinate system can be found in the studies by Bateman (1929), Herivel (1955), Lin (1963). A further description of this theory was made later by Seliger and Whitham (1968). The Euler equations can be derived directly from a generalized Bateman's variational principle. The Eulerian type of variational principle, on the other hand, can only be obtained by adding the appropriate constraints to the Lagrangian density of Hamilton's Principle (Seliger and Whitham, 1968; Serrin, 1959). It leads to a Clebsch transformation of velocity vector via a potential-like variable and Lagrangian multipliers known as Clebsch (1859) variables. Application of such variational formulation for solving compressible Euler equations has been presented previously by Ecer and his colleagues (Akay and Ecer, 1983, 1984, 1986; Ecer and Akay, 1983; Ecer *et al.*, 1984a, b). The extension to the viscous flow problem has been attempted and implemented numerically by the present author (Ecer *et al.*, 1986).

In this paper, the concept of the Bateman variational principle for compressible Euler equations is used to study the car aerodynamics. The finite element solutions were computed by using a CRAY X-MP super computer at National Taiwan University.

II. Variational Principle

The governing equations for describing three-dimensional, compressible Euler flows at steady state are as follows (Sheu, 1988):

Continuity equation:

$$(\rho u_j),_j = 0 \qquad (j = 1 \sim 3) \tag{1}$$

Momentum equation:

$$\rho \frac{Du_i}{Dt} = -p,_i \tag{2}$$

Energy equation:

$$(\rho s u_j),_j = 0 \tag{3}$$

The equation of state for perfect gas can be written as

$$p = \rho RT = K\rho^{\gamma} \exp((\gamma - 1)\frac{s}{R})$$
$$K = \frac{p_*}{\rho_*{}^{\gamma}} \exp(-(\gamma - 1)\frac{s_*}{R}) \tag{4}$$

where * denotes the reference condition.

The variational functional corresponding to equations (1–4) can be written as follows:

$$\pi = \int_\Omega [\frac{1}{2} \rho u_j u_j - \rho(E(\rho,s) - H_0) + \phi(\rho u_j),_j + \eta(\rho s u_j),_j] \, d\Omega - \int_{\tau_\phi} \phi(\rho u_j n_j + f)\, d\tau - \int_{\tau_\eta} \eta(\rho s u_j n_j + g)\, d\tau \quad (5)$$

where the Neumann boundary conditions

$$\rho u_j n_j + f = 0 \quad (6)$$

$$\rho s u_j n_j + g = 0 \quad (7)$$

are specified on the surfaces. ϕ, η in (5) are the Lagrangian multipliers added to the original Lagrangian density.

By applying the variational principle to functional (5), $\delta\pi = 0$, and performing integration by parts, the surface integrals disappear. This leads to a Clebsch transformation of velocity vector for an arbitrary δu_j:

$$u_j = \phi,_j + s\eta,_j \quad (8)$$

Equation (8) is the so-called Clebsch transformation for inviscid flows (Bateman, 1929; Herivel, 1955; Lin, 1963; Serrin, 1959). The above velocity decomposition is identical to that derived from the dual-potential formulation (Akay and Ecer, 1984, 1986; Chaderjian and Steger, 1985; Chang and Ademczyk, 1983; Ecer *et al.,* 1984a, b, 1986; Grossman, 1983; Hafez and Lovell, 1983; Sheu, 1988).

A set of governing equations, (1), (3), and equivalent momentum equations

$$\rho u_j \eta,_j = -\frac{p}{R} \quad (9)$$

are derived by considering the variations with respect to arbitrary $\delta\phi$, $\delta\eta$ and δs. Taking variation of the functional with respect to density ρ yields the expression for stagnation enthalpy:

$$H = H_0 \quad (10)$$

This indicates that constant stagnation enthalpy is maintained along the streamline.

The vorticity vector ω, and density ρ can be explicitly written in terms of primary variables by

$$\underline{\omega} = \nabla s \times \nabla \eta \quad (11)$$

and

$$\rho = \left(\frac{\gamma - 1}{K\gamma} h\right)^{\frac{1}{\gamma - 1}} \exp\left(-\frac{s}{R}\right) \quad (12)$$

By substituting the relations shown above into the original functional in equation (5), one can obtain the generalized variational functional using mass and entropy fluxes as natural boundary conditions:

$$\pi = \int_\Omega p \, d\Omega - \int_{\tau_\phi} \phi f \, d\tau - \int_{\tau_\eta} \eta g \, d\eta \quad (13)$$

The present variational functional can be verified in Ecer *et al.* (1986) and Sheu (1988) by showing the equivalence between two sets of equations, (1), (2), (3) and (1), (3), (8), (9). Thus one can employ either the original variational functional (5) with appropriate constraint conditiions or the generalized variational functional (13) to analyze the Euler flows.

III. Finite Element Discretization

The differential equations (1), (3), (8), (9) are cast to the second-order form by multiplying a convection operator on both sides of (3) and (9). This operation not only provides the symmetric matrices but also adds artificial viscosity so that stability can be enhanced (Akay and Ecer, 1983). The steady state solutions are computed via a time marching procedure. The relaxation solution algorithm is made by introducing a relaxation factor ω to the primary variable A as:

$$\nabla A^{\text{new}} = \nabla A^{\text{old}} + \frac{\Delta t}{\omega} \nabla A^{\text{old}},_t \quad (14)$$

The resulting pseudo-unsteady differential equations are then discretized by the isoparametric finite element method (Akay and Ecer, 1983). This leads to

$$\underline{A}^n \, \underline{\dot{X}}^n = \underline{R}^n \quad (15)$$

where

$$\underline{\dot{X}}^n = \begin{bmatrix} \underline{\dot{\phi}}^n \\ \underline{\dot{\eta}}^n \\ \underline{\dot{s}}^n \end{bmatrix} = \frac{1}{\omega} \begin{bmatrix} \underline{\phi}^{n+1} - \underline{\phi}^n \\ \underline{\eta}^{n+1} - \underline{\eta}^n \\ \underline{s}^{n+1} - \underline{s}^n \end{bmatrix}$$

$$\underline{A} = \begin{bmatrix} \underline{K}^n_{\phi\phi} & \underline{K}^n_{\phi\eta} & \underline{K}^n_{\phi s} \\ \underline{0} & \underline{K}^n_{s\eta} + \underline{K}^n_{da} & \underline{0} \\ \underline{0} & \underline{0} & \underline{K}^n_{s\eta} + \underline{K}^n_{da} \end{bmatrix}$$

$$\underline{R}^n = \begin{bmatrix} \underline{f}_\phi^n - \underline{K}_{\phi\phi}^n \ \underline{\phi}^n - \underline{K}_{\phi\eta}^n \ \eta^n \\ \underline{G}_s^n - \underline{K}_{s\eta}^n \ \underline{\eta}^n \\ - \underline{K}_{s\eta}^n \ \underline{s}^n \end{bmatrix}$$

$$\underline{K}_{\phi\phi} = \sum_e \int_{\Omega^e} \rho \ \underline{N}_j \ \underline{N}^T_{,i} \ d\Omega$$

$$\underline{K}_{\phi\eta} = \sum_e \int_{\Omega^e} \rho s \ \underline{N}_{,i} \ \underline{N}^T_{,j} \ d\Omega$$

$$\underline{K}_{\phi s} = \sum_e \int_{\Omega^e} \rho \eta_{,i} \ \underline{N}_{,i} \ \underline{N}^T_{,j} \ d\Omega$$

$$\underline{K}_{\phi\eta} = \sum_e \int_{\Omega^e} \rho u_i \ \underline{N}_j \ (\rho u_j \ \underline{N}_{,j})^T \ d\Omega$$

$$\underline{G}_s = \sum_e \int_{\Omega^e} -\frac{p}{R} \ (\rho u_i \ \underline{N}_{,i}) \ d\Omega$$

$$\underline{f}_\phi = \sum_e \int_{\tau_\phi} (\rho u_i \ \underline{N}_{,i}) \ d\Omega$$

It is suggested to add the damping term

$$K_{da} = (\text{damping factor}) \times \int_{\Omega^e} N_{,i} \ N_{,i} \ d\Omega$$

to equation (15) to prevent the possible appearance of numerical disturbances (Akay and Ecer, 1983). The addition of this damping term does not change the solution when steady state is reached.

The solutions of (15) can be obtained in an uncoupled sequence by first calculating $\dot{\underline{\eta}}$ and $\dot{\underline{s}}$:

$$\dot{\underline{\eta}}^n = (\underline{K}_{s\eta}^n + \underline{K}_{da}^n)^{-1} (\underline{G}_s^n - \underline{K}_{s\eta}^n \ \underline{\eta}^n) \tag{16}$$

$$\dot{\underline{s}}^n = (\underline{K}_{s\eta}^n + \underline{K}_{da}^n)^{-1} (-\underline{K}_{s\eta}^n \ \underline{s}^n) \tag{17}$$

The solution of $\dot{\phi}$ is then computed. The solution is advanced from time $n\Delta t$ to $(n+1)\Delta t$ until the steady state is reached.

IV. Results and Discussion

The developed variational finite element computer code has been validated by comparing the analytic data for solving an inviscid fluid over a cylinder (Huang, 1990). A practical application of predicting an inviscid flowfield around a car of simple geometry is attempted in the present study. Consider a car at 100 Km/hr in a flow condition of 1 atm (103325 N/m^2), 25°C. The investigated flow domain and its finite element grid is shown in figure 1. The shape and the grid points on the surfaces of the car are shown in figure 2. Figure 3 shows the computed velocity vector at the symmetric plane of the flowfield, and figure 4 shows the velocity vector on the surfaces of the car. Figure 5 shows the velocity vectors along the planes at x = 0.0 m, 2.0 m, 3.0 m, 6.0 m, and 10.0 m. The velocity vectors along the planes at y = 0.5 m, 1.5 m, and 2.5 m, and the velocity vectors along the planes at z = 0.0 m, and 1.0 m are shown, respectively, in figures 6 and 7. The pressure contours on the surface of the car are shown in figure 8. The contours of Lagrangian multiplies ϕ and η on the symmetric plane are shown in figure 9 and figure 10. The convergence history for the primary

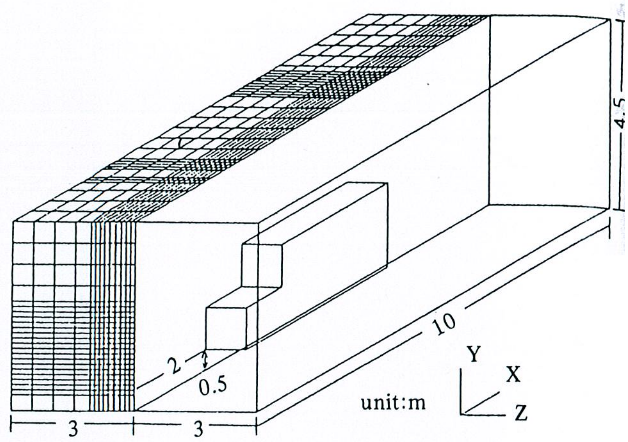

Fig. 1. Flow domain and its finite element grids.

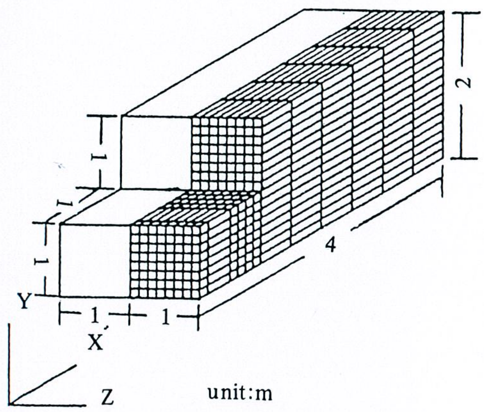

Fig. 2. The shape and the grid points in the car.

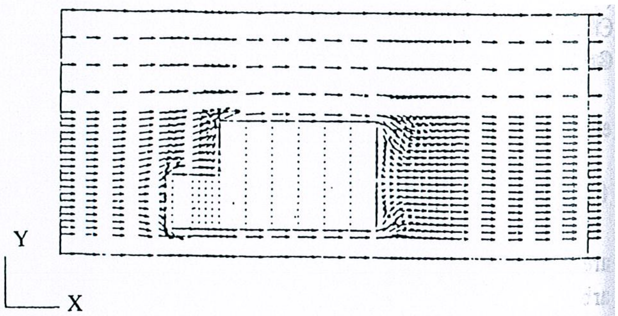

Fig. 3. The computed velocity vector at the symmetric plane of the flow field.

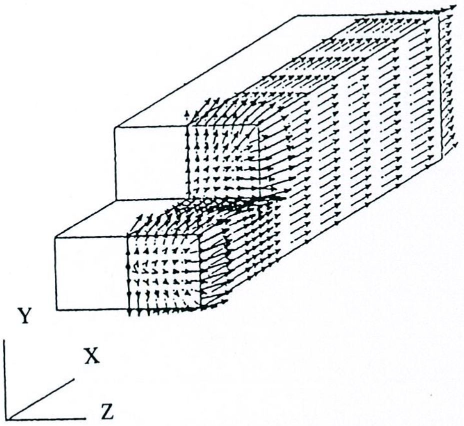

Fig. 4. The velocity vector on the surfaces of the car.

variables is shown in figure 11. The computed velocity field agrees reasonably with what we expect for the inviscid case. Further comparison with available data will be attempted later.

V. Conclusion

A variational finite element formulation for predicting inviscid three-dimensional flow has been developed. The resulting Clebsch formulation may provide an alternative way to solve a complex multi-dimensional inviscid flow problem. Numerical prediction of inviscid air flows over a rectangular car has been conducted using the present computer code.

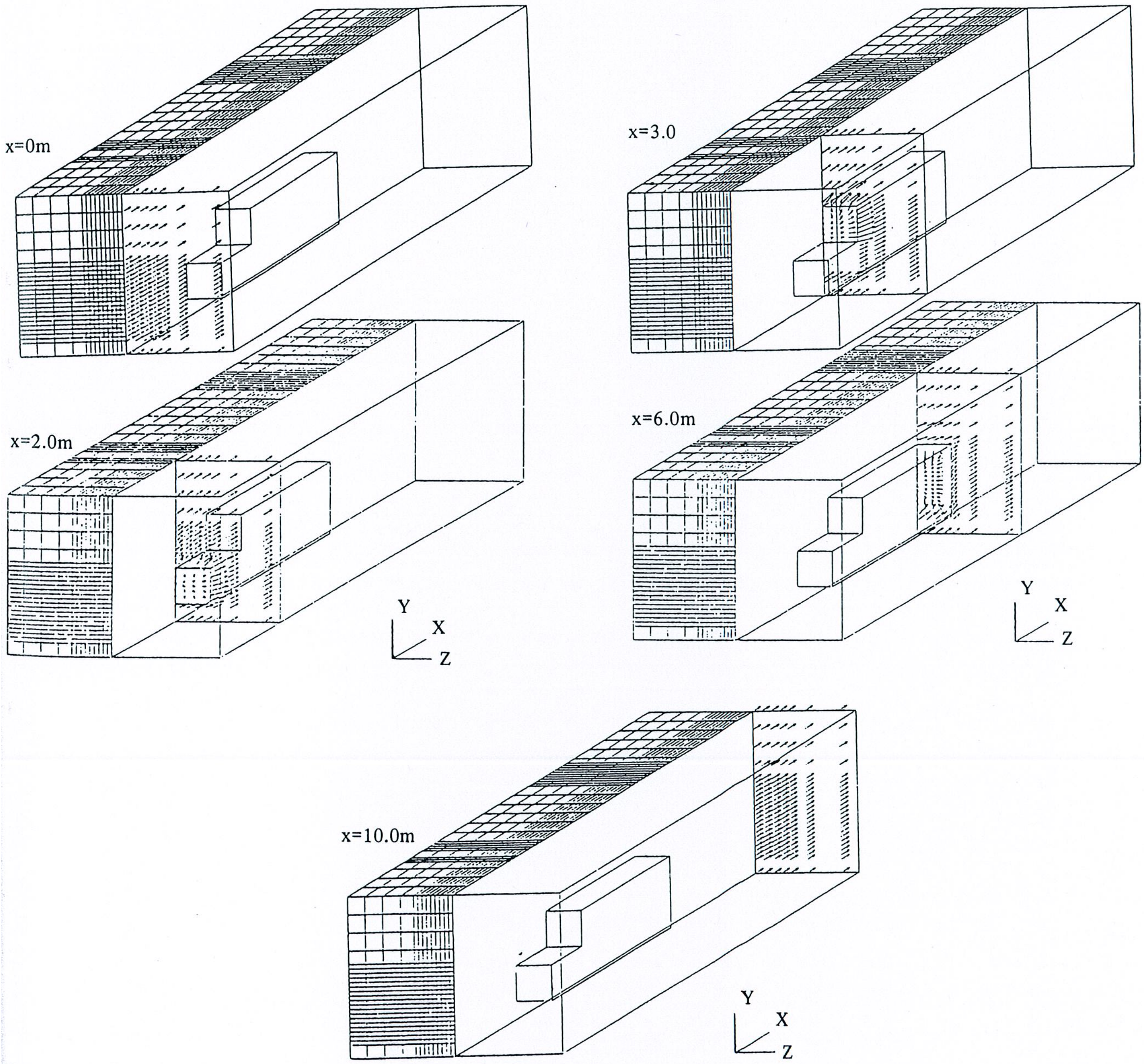

Fig. 5. The velocity vectors along the planes at x = 0.0m, 2.0m, 3.0m, 6.0m, and 10.0m.

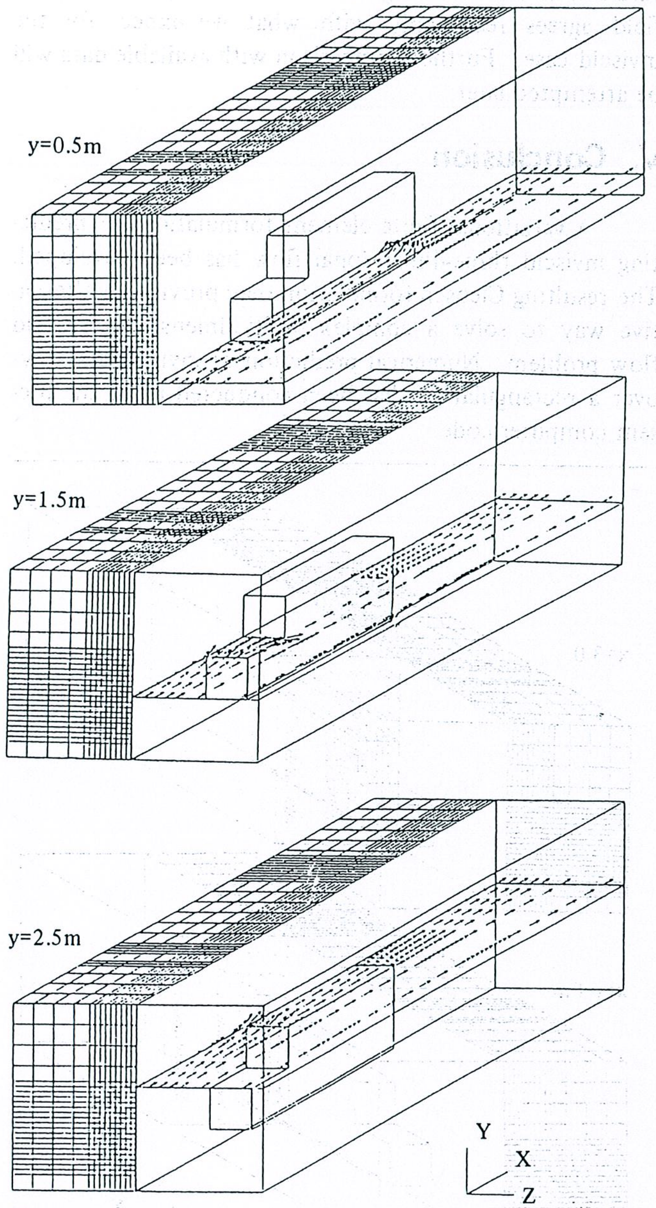

Fig. 6. The velocity vectors along the planes at y = 0.5m, 1.5m, and 2.5m.

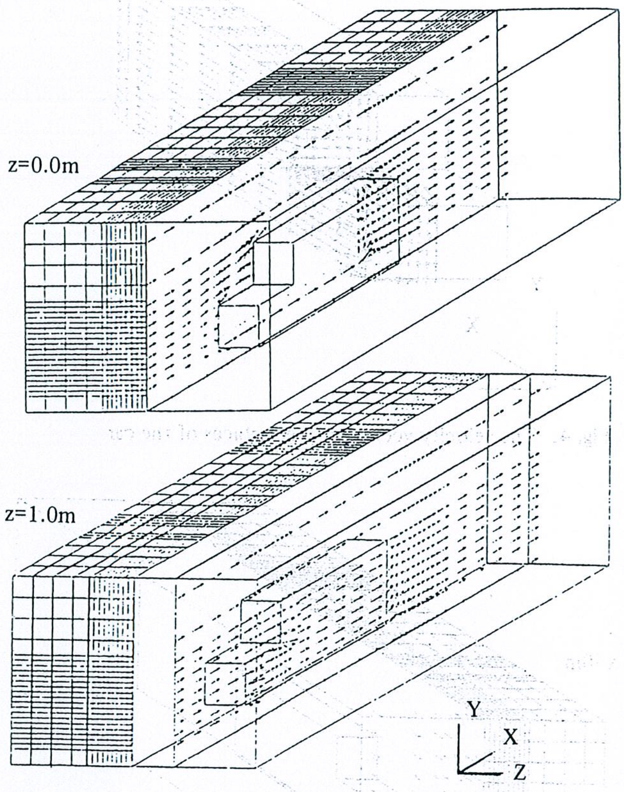

Fig. 7. The velocity vectors along the planes at z = 0.0m, and 1.0m.

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Nomenclature

E : internal energy per unit mass
f : normal mass flux
g : normal entropy flux
H : stagnation enthalpy
H_0 : reference constant for the internal energy of a particle
$\underline{K}$: coefficient matrix of the finite element equation
n_i : components of the normal vector
p : static pressure
R : gas constant
s : entropy
T : temperature
τ : boundary domain
γ : ratio of specific heats
δ : variational operator
η : Lagrangian multiplier
π : variational functional
ρ : mass density
u_i : velocity component in the i direction
h : enthalpy
$\underline{\omega}$: vorticity vector
$\frac{D}{Dt}$: total derivative
$,j$: derivative with respect to j
ω : relaxation factor

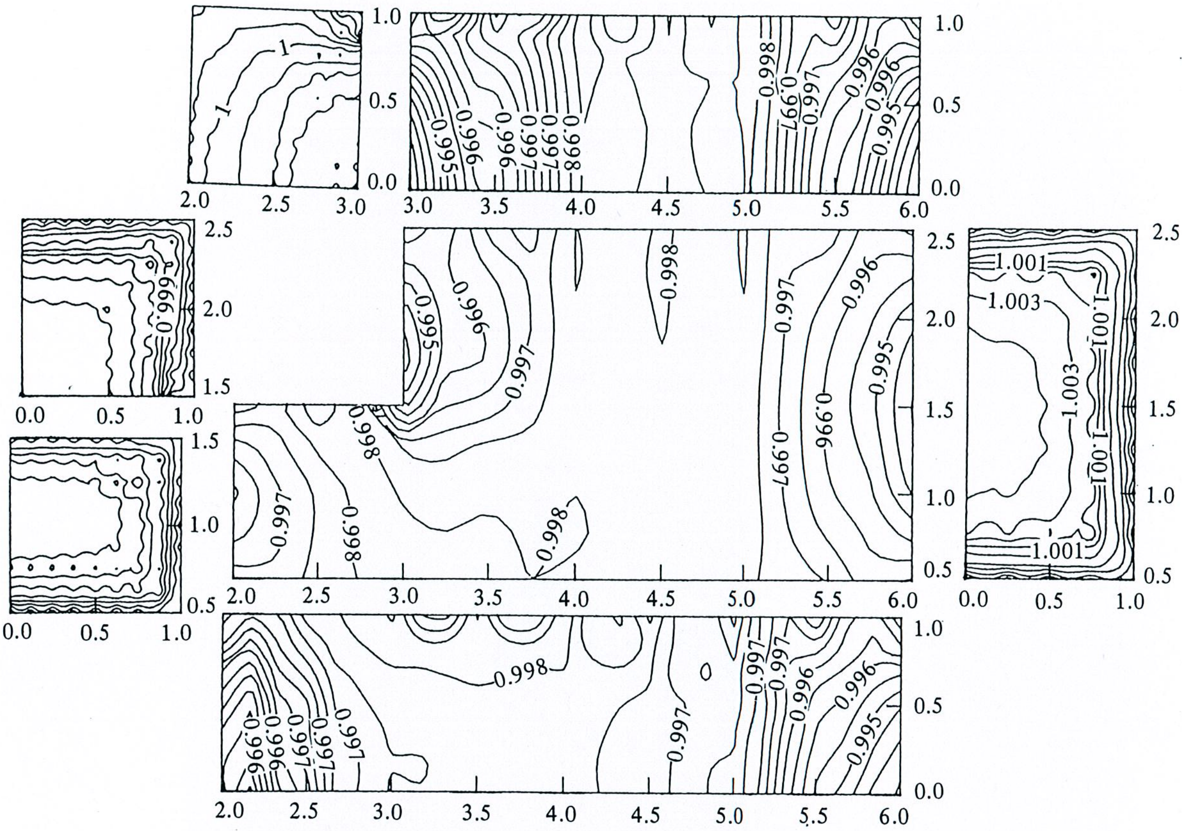

Fig. 8. The pressure contours in the surface of the surface of the car.

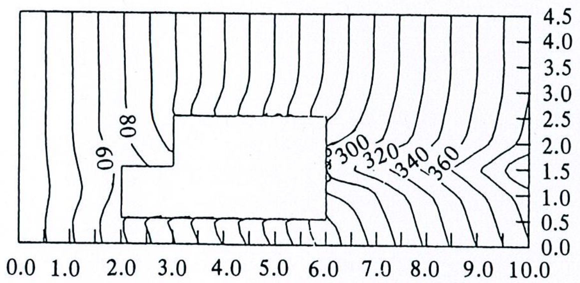

Fig. 9. The contours of Lagrangian multiplies ϕ on the symmetric plane.

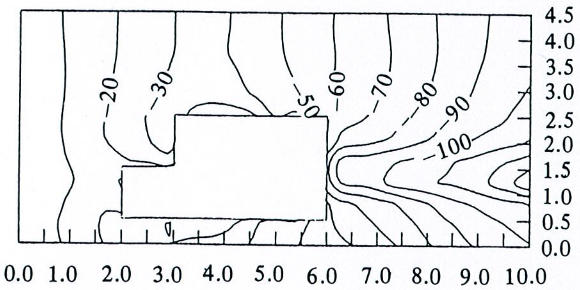

Fig. 10. The contours of Lagrangian multiplies η on the symmetric plane.

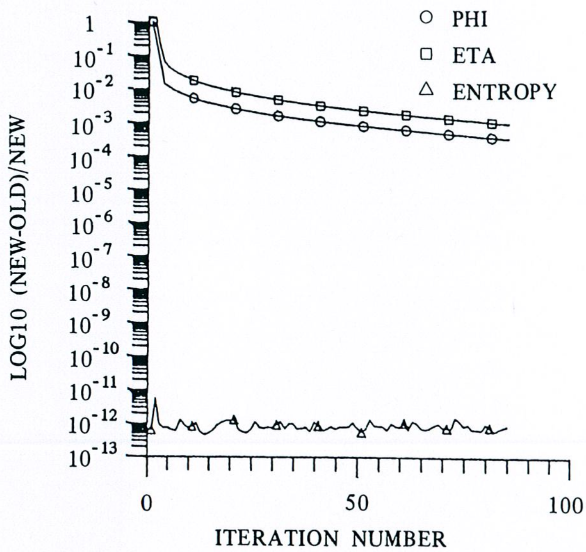

Fig. 11. The convergence history for the primary variables.

$\underline{N}$: shape function

K : a constant in Eq. (4) is computed from reference conditions

$*$: reference condition

$\cdot$: defined in equation (15)

Ω : physical domain

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有限元素方法解析三維無黏性流場

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摘 要

本文利用以能量觀點的變分方法分析三維可壓縮無黏性的穩定流場。利用變分原理可以導出克式速度轉換式和以熵及拉氏乘子爲主要變數的變分方程組。該偏微分方程組和以原始變數爲主要變數的尤拉運動方程式是全等的，並可用來分析勢流和尤拉流場。利用有限元素方式將變分方程組離散。利用具有鬆性的計算程序解析流經一三維物體的低馬赫的周圍流場。