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On the Development of Generalized Variational Functional for Compressible Navier-Stokes Flows

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建立一套解析壓縮黏性流場之變分函數

許文翰*

摘　要

本文發展一套變分方法分析非穩態的三維壓縮黏性流場。藉此速度向量可以 Clebsch 轉換表示。控制方程式可以寫成 Lagrangian multiplier 及火商的表示式，而且可證明它是和以原始變數的控制方程組全等。此一變分方法提供了統一的方法解析勢流、尤拉流、及那維式史多客流。

關鍵詞： 變分函數，壓縮，那維式史多客流，拉氏乘子。

Abstract

A variational method based on energy formulation for analyzing time dependent, three-dimensional, compressible, and viscous flow-field is developed. A Clebsch tranformation of the velocity vector and a set of governing equations in terms of Lagrangian multipliers and entropy are derived. This formulation is equivalent to that of Navier-Stokes equations in terms of primitive variables. It provides a unified solution scheme for potential, Euler and Navier-Stokes equations if different levels of flow simplification are made.

Keywords: variational functional, compressible, Navier-Stokes flow, Lagrangian multiplier.

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Introduction

The development of variational principle in fluid dynamics is one of the important issues in the classical dynamics. It is known that the direct variational formulation for a problem, written in a self-adjoint differential operator form, can be derived over the Lagrangian coordinate system [1]. However, additional efforts must be made in obtaining a variational principle for fluid flow in Eulerian description.

A valuable source of constructing a variational principle for inviscid flow problems over the Eulerian coordinate system can be found in the works of Bateman [2], Herivel [3], Lin [4], and Serrin [5]. A further description of this theory was made later by Seliger and Whitham [6]. From their efforts, a set of Euler equations can be derived directly from a generalized Bateman's variational principle. An Eulerian variational principle is obtained by adding the physically appropriate constraints to the Lagrangian density of Hamiliton's Principle [5,6]. It leads to a Clebsch transformation of the velocity vector in terms of potential-like variables and Lagrangian multipliers known as Clebsch variables [7]. Numerical application to compressible Euler equations based on this formulation have been presented earlier by A. Ecer and his colleagues with good agreement.

In this paper, the concept of developing a variational principle for compressible Navier-Stokes equations is presented. This formulation provides potential and Euler formulations, reducing to Bateman's Principle, as the special case. The verification is made by showing the derived set of equations is equivalent to the conventional momentum equations in primitive variable form.

1. Variational function

The governing equations in Eulerian description for describing three-dimensional, compressible Navier-Stokes flows are:
Continuity equation

$$\rho_{,t}+(\rho u_j)_{,j}=0 \ (j=1\sim3) \tag{1.1}$$

Momentum equations

$$\rho\frac{Du_i}{Dt}=-P_{,i}+[2\mu(e_{ij}-\frac{1}{3}e_{kk}\ \delta_{ij})]_{,j} \tag{1.2}$$

Energy equation

$$(\rho S)_{,t}+(\rho Su_j)_{,j}=\frac{\Phi}{T} \tag{1.3}$$

Equation of state for perfect gas

$$P=\rho RT=K\rho^{\gamma}\ \exp((\gamma-1)\frac{S}{R}) \tag{1.4}$$

where

$$K=\frac{P*}{\rho*^{\gamma}}\ \exp(-(\gamma-1)\frac{S*}{R} \tag{1.5}$$

and * denotes the reference condition.

The viscous dissipation function Φ in equation(1.3)is

$$\Phi=2\mu(e_{ij}e_{ij}-\frac{1}{3}(u_{k,k})^2) \tag{1.6}$$

where e_{ij} is the rate of shear.

$$e_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}) \tag{1.7}$$

μ in equation (1.6) is called the laminar viscosity which is modeled by Sutherland's Law [8].

As far as the solution of a boundary value problem by using a variational method is considered, a corresponding variational functional of the given differential equations (1.1)–(1.7) is required. Extensive attempts in obtaining the variational formulations are mainly to inviscid flow problems by imposing a set of constrained conditions using Lagrangian multipliers [1–6]. In the case of viscous flow, Finlayson [9,10] indicated that there is no variational functional for Navier-Stokes equations in terms of primitive variables. The objective of this paper is to develop an Eulerian variational principle to analyze Navier-Stokes flow. The fundamental difficulty in developing an Eulerian variational principle can be overcome by introducing appropriate constraints on the variational functional.

In the present formulation for compressible and viscous flows, the continuity equation (1.1), conservation of entropy equation (1.3), and the rate of shear defined by (1.7) are employed as the constraints to specify the motions of fluid particles [11,12]. The variational functional using (1.1), (1.3), and (1.7) as constraints is expressed as:

$$\begin{aligned}\Pi = \int_\Omega [\frac{1}{2}\rho u_j u_j - \rho(E(\rho, S)-H_o)+\emptyset(\rho_{,t}+(\rho u_j)_{,j})+\eta((\rho S)_{,t}+(\rho S u_j)_{,j}\\ -\frac{\Phi}{T})+k_{ij}(e_{ij}-\frac{1}{2}(u_{i,j}+u_{j,i}))]\,d\Omega\,dt-\int_{\Gamma_\emptyset}\emptyset(\rho u_j n_j+f)d\Gamma dt\\ -\int_{\Gamma_\eta}\eta(\rho S u_j n_j+g)d\Gamma dt+\int_{\Gamma_v}k_{ij}n_j(u_i+h_i)d\Gamma dt\end{aligned} \tag{1.8}$$

where the Neumann boundary conditions

$$\rho u_j n_j+f=0 \tag{1.9}$$

$$\rho S u_j n_j+g=0 \tag{1.10}$$

are specified on the surfaces, with an outward normal n_j, to completely define the transport properties. In addition, the Dirichlet boundary condition

$$u_i+h_i=0 \tag{1.11}$$

is specified as an essential condition for velocity components at the wall. In the above functional (1.8) ,$\emptyset$, η ,k_{ij} are called Lagrangian multipliers which correspond to the constrained conditions added to the original Lagrangian density.

2. Variational Principle

If one applies variational principle on functional (1.8), $\delta\Pi=0$,and integration by parts,the surface integrals disappear. It leads to a Clebsch transformation of velocity vector for arbitrary δu_j.

$$u_j=\emptyset_{,j}+S\eta_{,j}-\frac{1}{\rho}k_{ij,i} \tag{2.1}$$

The expression of velocity components in (2.1) reduces to Clebsch transformation for inviscid flows [2,3,5].It is identical to the approach of dual-potential formulation [13−16] for inviscid rotational flows and scalar-vector potential formulation [17,18] for Navier-Stokes flows.

A set governing equations (1.1),(1.3) and equivalent momentum equations

$$(\rho\eta)_{,t}+\rho u_i\eta_{,i}=-\frac{P}{R} \tag{2.2}$$

for describing the equations of motion is derived by considering the variations of arbitrary $\delta\emptyset$, $\delta\eta$ and δS respectively.It is noted that the transport equations (1.1),(1.3) were rederived which are introduced as constraints on the variational functional.

The variation of functional with respect to density ρ yields the expression for stagnation enthalpy.

$$H=H_o-\frac{1}{\rho}u_ik_{ij,j}-\phi_{,t}-S\eta_{,t} \tag{2.3}$$

It reduces to the constant stagnation enthalpy for inviscid flow along the streamline. In general, the stagnation enthalpy is not necessarily a constant due to its unsteady and viscous effects. The viscous stresses will act on the boundaries of element and do work to accelerate and deform the fluid element such that the kinetic and internal energies will change accordingly [19]. Similarly, by taking variations with respect to k_{ij} and e_{ij}, one can rederive (1.7) and express Lagrangian multipliers k_{ij}

$$k_{ij}=\frac{4\mu\eta}{T}(e_{ij}-\frac{1}{3}e_{kk}\delta_{ij}) \tag{2.4}$$

in terms μ η, and e_{ij}.

The values of remaining flow properties such as vorticity vector ω and density ρ can be evaluated directly from the solutions of primary variables by using

$$\omega=\nabla S\times\nabla\eta-\nabla\times(\frac{1}{\rho}k_{ij,j}) \tag{2.5}$$

and

$$\rho=(\frac{\gamma-1}{K\gamma}h)^{\frac{1}{\gamma-1}}\exp(-\frac{S}{R}) \tag{2.6}$$

By substituting the relations shown above into the original functional in equation (1.8), one can obtain the generalized variational functional.

$$\Pi=\int_\Omega p-\frac{1}{2}k_{ij}e_{ij}d\Omega dt-\int_{\Gamma_\phi}\phi fd\Gamma dt-\int_{\Gamma_\eta}\eta gd\Gamma dt \tag{2.7}$$

As can be seen from above equation, only the normal fluxes of mass and entropy have to be specified as natural boundary conditions. The well known Bateman's Principle for inviscid flows without k_ij turns out to be the special case [2,3,5,6].

3. Verification of Mathematical Model

The validity of the present constrained variational functional can be verified by showing the equivalent relations between the set of equations (1.1), (1.2), (1.3) and that of (1.1), (1.3), (2.1), (2.2), and (2.4).

By substituting the Clebsch transformation for velocity into the left hand side of equation (1.2), on can obtain

$$\rho\frac{Du_i}{Dt}+p_{,i}=\rho H_{,i}+\frac{\Phi}{T}\eta_{,i}-[u_j(k_{mj,m})_{,i}-\frac{D}{Dt}(k_{mi,j})] \qquad (3.1)$$

by the use of equations (1.3), (2.2), and thermodynamic relation.

$$TS_{,j}=E_{,j}+P(\frac{1}{\rho})_{,j}$$

Multiplying both sides of equations (3.1) and (1.2) by u_i, the equivalency can be achieved since

$$\rho\, u_i H_{,i}+\frac{\Phi}{T}u_i\,\eta_{,i}-u_i(u_j(k_{mj,m})_{,i}-\frac{D}{Dt}(k_{mi,m}))$$
$$=u_i(2\mu(e_{ij}-\frac{1}{3}e_{kk}\ \delta_{ij}))_{,j}$$

is derived by the use of conservation of stagnation enthalpy:

$$\rho\frac{DH}{Dt}=(2\mu\, u_i(e_{ij}-\frac{1}{3}e_{kk}\ \delta_{ij}))_{,j}$$

This verification provides a complete formulation of mathematical model. One can thus employ either the orginal variational (1.8) with proper constraint conditions or the generalized variational functional (2.7) to analyze Navier-Stokes flows.

The application of weighted residual finite element method to the present variational formulation for analyzing compressible Navier-Stokes flows at steady state has been made [20]. The computed results of entrance flow problem showed good agreements with those by theoretical and finite difference approaches(Fig.1).

4. Conclusion

A set of equations for describing Navier-Stokes flowfield is equivalent to that of primitive variables form. The extension of Bateman's Principle for inviscid flows to viscous counterparts is made directly. The derived set of equations in terms of Clebsch variables provides a new alternative to simulate viscous flowfield. One can compare the solutions of potential, Euler and Navier-Stokes flows by using the same numerical procedure. Such a generalized framework provides the computational efficiency for the analysis of Navier-Stokes flow by starting with the solution of inviscid flow. Considerable computing time may be reduced if different levels of flow simplification, suitable for developing block-structured algorithm, are made in the case of multi-dimensional flow problems.

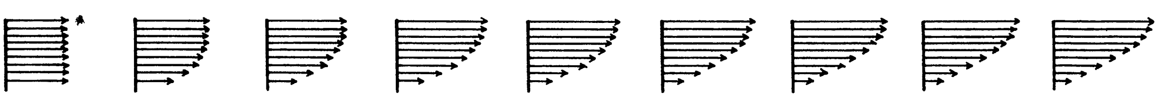

Fig. 1(a) Velocity Vectors at Different Sections of a Straight Duct (Inlet Re=153,△x=0.24m)

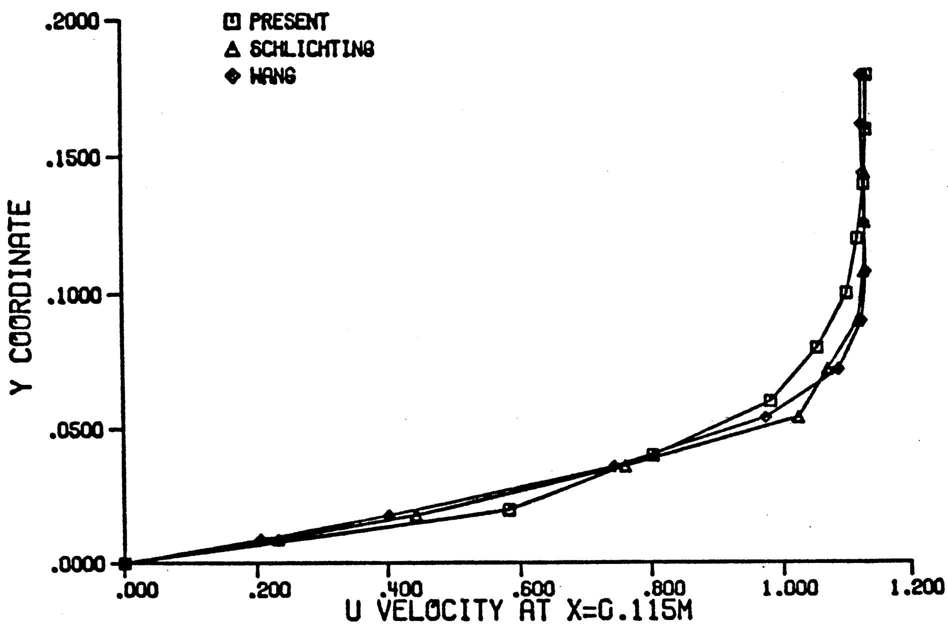

Fig. 1(b) Comparisons of U Velocity with W.L. Wang and Schlichting at Location x=0.115 m

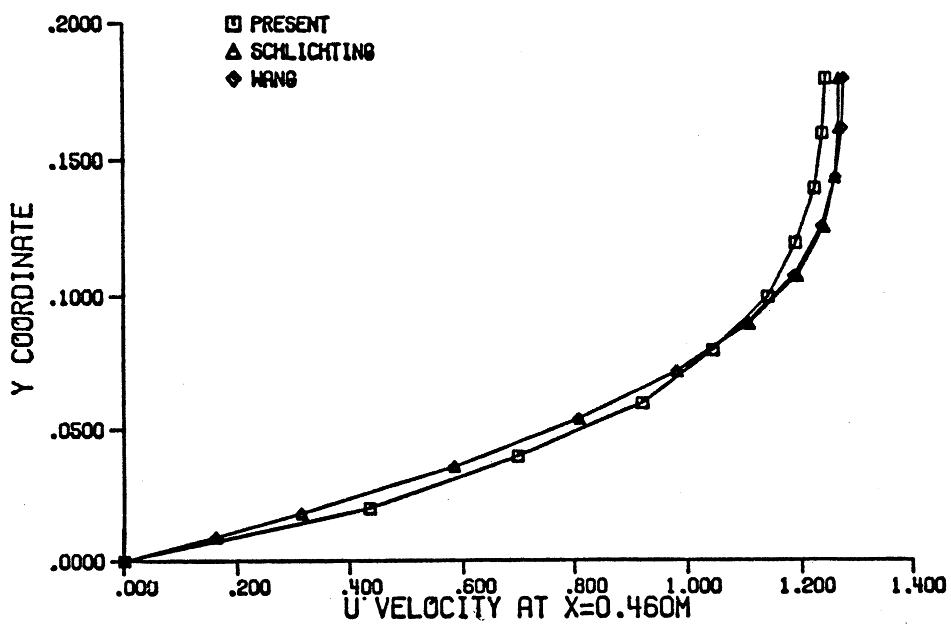

Fig. 1(c) Comparisons of U Velocity with W.L. Wang and Schlichting at Location x=0.46 m

Nomenclature

E : internal energy per unit mass
e_{ij} : rate of strain tensor
f : normal mass flux
g : normal entropy
H : stagnation enthalpy

o : reference constant for internal energy of a particle
n_i : components of normal vector
p : static pressure
R : gas constant
S : entropy
T : temperature
Γ : boundary domain
γ : ratio of specific heats
δ : variational operator
η : Lagrangian multiplier
μ : laminar viscosity
Π : variational functional
ρ : mass density
Φ : viscous dissipation function
u_i : velocity component in i direction
k_{ij} : Lagrangian multiplier
h : enthalpy
ω : vorticity vector
$\frac{D}{Dt}$: substantial derivative
,j : derivative with respect to j
δ_{ij} : Kronecker's delta function
ϕ : Lagrangian multiplier
S : entropy

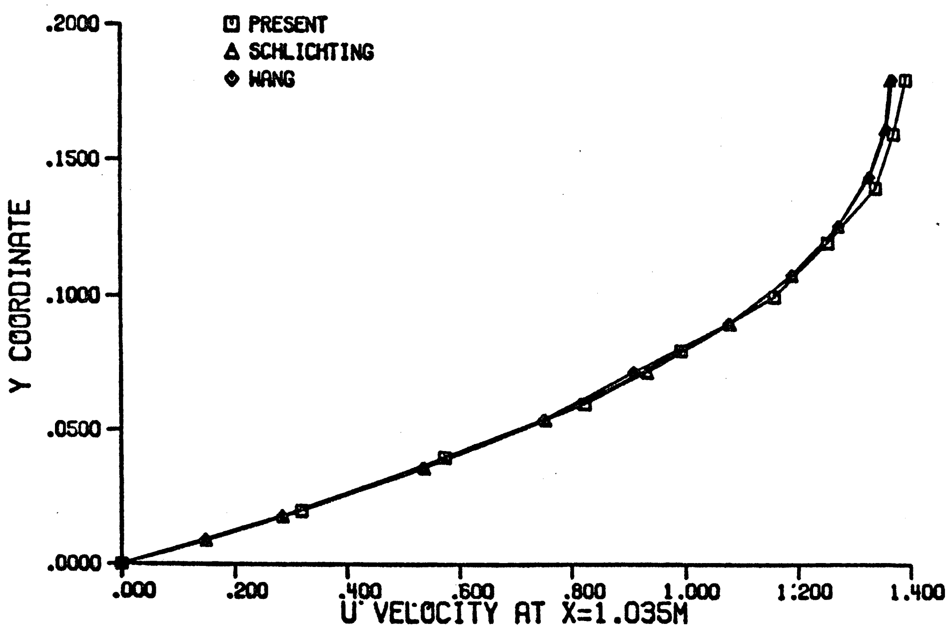

Fig. 1(d) Comparisons of U Velocity with W.L. Wang and Schlichting at Location x=1.035 m

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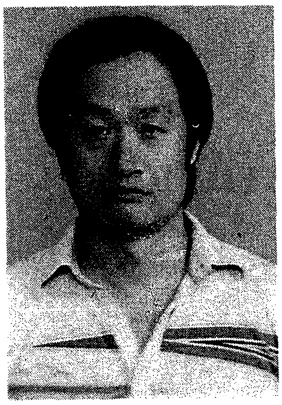

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