A Variational Finite Element Method for Compressible Navier–Stokes Flows

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ABSTRACT

A variational method is developed for analyzing three–dimensional steady, compressible and viscous flow–field starting with the energy formulation. A Clebsch transformation of the velocity vector and a set of governing equations in terms of Lagrangian multipliers and entropy are derived. This mathematical model is equivalent to the classic full Navier–Stokes equations in terms of primitive variables. It provides an unified solution scheme for potential, Euler and Navier–Stokes flow equations if different levels of flow simplification are made. The isoparametric finite element approximation and a relaxation solution scheme are employed to obtain the solutions at steady–state in an uncoupled sequence.

A computer code is developed and verified by comparing the computed solutions with the available theoretical results of developing entrance channel flow. A convergent channel flow problem is also investigated.

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Introduction

The development of variational principle in fluid mechanics is one of the important issues in the classical dynamics. It is known that the direct variational formulation of a problem, written in self–adjoint differential operator form, can be derived over the Lagrangian coordinate system[1]. Additional efforts must be made for obtaining a variational principle of fluid dynamics equations in Eulerian description.

A valuable source of variational principle for inviscid flow problems over the Eulerian coordinate system can be found in the classical works of Bateman[2], Herivel[3], Lin[4] and Serrin[5]. A further description of this theory was made later by Seliger and Whitham[6]. From these efforts, a set of Euler equations can be derived directly from a generalized Bateman's variational principle. An Eulerian variational principle is obtained by adding physically appropriate constraints to the Lagrangian density of Hamiliton's Principle[5,6]. It leads to a Clebsch transformation of the velocity vector in terms of potential–like variable and Lagrangian multipliers known as Clebsch variables[7]. Numerical implementations of variational formulation for compressible Euler equations have been presented earlier by Ecer and his colleagues[8–13].

In this paper, the concept of developing a variational principle for compressible Navier–Stokes equations is presented. This formulation provides potential and Euler formulations, reducing to Bateman's principle, as the special cases. The verification of this variational principle is made by showing: (1) the derived set of equations is equivalent to the conventional momentum equations in primitive variables form, (2) the solutions by finite element approximation are compared with the analytic solutions of developing channel flows.

Formulation of the variational principle

The governing equations in Eulerian description for describing three–dimensional, compressible Navier–Stokes flows at steady state are[17]:

Continuity equation

$$(\rho u_j)_{,j} = 0 \quad (j = 1 \sim 3) \tag{1}$$

Momentum equations

$$\rho \frac{Du_i}{Dt} = -p_{,i} + [2\mu(e_{ij} - \frac{1}{3} e_{kk}\delta_{ij})]_{,j} \tag{2}$$

Energy equation, satisfing Stokes' hypothesis, in terms of entropy

$$(\rho S u_j)_{,j} = \frac{\Phi}{T} \tag{3}$$

and equation of state for perfect gas

$$p = \rho R T = K \rho^{\gamma} \exp\left((\gamma - 1) \frac{S}{R} \right)$$

$$K = \frac{p_*}{\rho_*^{\gamma}} \exp\left(-(\gamma - 1) \frac{S_*}{R} \right) \tag{4}$$

where $_*$ denotes the reference conditions.

The viscous dissipation function Φ in equation (3) is

$$\Phi = 2 \mu \left(e_{ij} e_{ij} - \frac{1}{3} (u_{k,k})^2 \right) \tag{5}$$

where e_{ij} is the rate of shear defined by

$$e_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \tag{6}$$

and μ is laminar viscosity modelled by Sutherland's Law[14] as

$$\mu(T) = \mu_o \left(\frac{T}{T_o} \right)^{1.5} \frac{T_o + S_1}{T + S_1} \tag{7}$$

where $S_1 = 110^{\circ}K$, $\mu_o = 0.16758 \times 10^{-4} \frac{N\text{—}S}{m^2}$

For solutions of boundary value problems by using a variational method, a equivalent variational form of the given differential equations (1)˜(4) is required Finlayson[15,16] indicated that there is no variational formulation for Navier–Stoke equations in terms of primitive variables. Over the years, attempts have been made to obtain the variational formulations mainly to inviscid problems by imposing a set o constrained conditions using Lagrangian multipliers[1–6]. The objective of this paper is to develop an Eulerian variational principle which describes the Navier–Stokes flow–field. Th fundamental difficulty in developing an Eulerian variational principle can be overcome b introducing the appropriate constraints on the variational functional.

In the present compressible and viscous flows, the equation of mass (1) conservation of entropy equation (3) and the rate of shear defined by (6) are employed as th constraints to specify the motions of fluid particles[17]. The variational functional is writte as:

$$\Pi = \int_\Omega \Big[\frac{1}{2}\rho u_j u_j - \rho (E(\rho, S) - H_o) + \phi(\rho u_j)_{,j} + \eta((\rho S u_j)_{,j} - \frac{\Phi}{T}) + k_{ij} (e_{ij} - \frac{1}{2}(u_{i,j} + u_{j,i})) \Big] d\Omega - \int_{\Gamma_\phi} \phi (\rho u_j n_j + f) d\Gamma - \int_{\Gamma_\eta} \eta (\rho S u_j n_j + g) d\Gamma + \int_{\Gamma_v} k_{ij} n_j (u_i + h_i) d\Gamma \tag{8}$$

where the Neumann boundary conditions

$$\rho u_j n_j + f = 0 \tag{9}$$

$$\rho S u_j n_j + g = 0 \tag{10}$$

are specified on the surfaces, with an outward normal n_j, to completely define the transport properties.

Dirichlet boundary condition

$$u_i + h_i = 0 \tag{11}$$

is specified as an essential condition for considering the velocity conditions at the wall. In the above functional (8), ϕ, η, k_{ij} are the Lagrangian multipliers corresponing to the constrained conditions added to the original Lagrangian density.

By applying the variational principle on functional (8), $\delta \Pi = 0$, and employing integration by parts, th

surface integrals disappear. It leads to a Clebsch transformation of velocity vector for arbitrary δu_j

$$u_j = \phi_{,j} + S \eta_{,j} - \frac{1}{\rho} k_{ij,i} \tag{12}$$

Equation (12) reduces to Clebsch transformation for inviscid flows[2,3,5]. It is identical to the approaches of dual–potential formulations[19–22] for inviscid rotational flows and scalar–vector potential formulations[23–25] for Navier–Stokes flows.

A set of governing equations (1), (3) and equivalent momentum equation

$$\rho u_i \eta_{,i} = - \frac{p}{R} \tag{13}$$

for describing the equations of motion is derived by considering the variations with respect to arbitrary $\delta\phi$, $\delta\eta$ and δS respectively. One can observe that the above variations provide the transport equations, which were specified as constraints on the variational form.

The variation of functional with respect to density ρ yields the expression for stagnation enthalpy

$$H = H_o - \frac{1}{\rho} u_i k_{ij,j} \tag{14}$$

which reduces to the constant stagnation enthalpy for inviscid flow along the streamline. In general, the stagnation enthalpy is not necessarily a constant. The viscous stresses will act on the boundaries of element and do work to accelerate and deform the fluid element such that the kinetic and internal energies will change accordingly[26].

At this point, by considering variations with respect to the remaining variables, equations (6) and expression for Lagrangian multipliers k_{ij}

$$k_{ij} = \frac{4 \mu \eta}{T} (e_{ij} - \frac{1}{3} e_{kk} \delta_{ij}) \tag{15}$$

can be derived from arbitrary δk_{ij} and δe_{ij} respectively.

The vorticity vector $\underline{\omega}$, density ρ can be explicitly written in terms of the primary variables by

$$\underline{\omega} = \underline{\nabla} S \times \underline{\nabla} \eta - \underline{\nabla} \times (\frac{1}{\rho} k_{ij,j}) \tag{16}$$

and

$$\rho = (\frac{\gamma - 1}{K \gamma} h)^{\frac{1}{\gamma - 1}} \exp (- \frac{S}{R}) \tag{17}$$

By substituting the relations shown above into the original functional in equation (8), one can obtain the generalized variational functional

$$\Pi = \int_{\Omega} p - \frac{1}{2} k_{ij} e_{ij} \, d\Omega - \int_{\Gamma_\phi} \phi f \, d\Gamma - \int_{\Gamma_\eta} \eta g \, d\Gamma \tag{18}$$

As can been seen from above equation, only the normal fluxes of mass and entropy have to be specified as natural boundary conditions. The well known Bateman's Principle for inviscid flows turns out to be the special case[2,3,5,6].

Verification of mathematical model

The validity of the present constrained variational functional can be verified by showing the equivalent relations between the set of equations (1), (2), (3) and (1), (12), (13), (15), (3).

By substituting the Clebsch transformation for velocity into the left hand side of equation (2), one can obtain

$$\rho \frac{D u_i}{D t} + p_{,i} = \rho H_{,i} + \frac{\Phi}{T} \eta_{,i} - [u_j (k_{mj,m})_{,i} - \frac{D}{Dt} (k_{mi,i})] \tag{19}$$

by the use of equations (3), (13), and thermodynamic relation

$$T S_{,j} = E_{,j} + p (\frac{1}{\rho})_{,j}$$

Multiplying both sides of equations (19) and (2) by u_i , the equivalency can be achieved since

$$\rho u_i H_{,i} + \frac{\Phi}{T} u_i \eta_{,i} - u_i (u_j (k_{mj,m})_{,i} - \frac{D}{Dt} (k_{mi,m}))$$
$$= u_i (2 \mu (e_{ij} - \frac{1}{3} e_{kk} \delta_{ij}))_{,j}$$

is obtained by the use of conservation of stagnation enthalpy.

This provides a complete formulation of mathematical model. One can employ either the original variational functional (8) with proper constraint conditions or the generalized variational functional (18) to analyze Navier–Stokes flows.

Development of finite element equations

The differential equations (1), (3), (12), (13) are a set of first–order equations for describing compressible and viscous flows. It can be cast in second–order forms in S and η respectively by multiplying both sides of (3) and (13) by a convection operator. This transformation not only provides symmetric matrices but also eliminates the use of artificial dissipative mechanism, such as upwinding, in obtaining numerical solutions.

A set of pseudo–unsteady equations is developed for the steady state solutions by time marching procedure. It is obtained by employing the relaxation scheme for primary variables A in the derived set of second–order equations

$$\underline{\nabla} A^{new} = \underline{\nabla} A^{old} + \frac{\Delta t}{\omega} \underline{\nabla} A^{old}_{,t}$$

where ω is a relaxation factor for the stable integration of nonlinear equations.

At this point, the weak variational form of pseudo–unsteady equations can be written as:

$$\int_{\Omega}\Big[\Big(\frac{-\Delta t}{\omega}(\rho\underline{\nabla}\phi_{,t}+\rho S\underline{\nabla}\eta_{,t}+\rho S_{,t}\underline{\nabla}\eta)$$
$$-(\rho\underline{\nabla}\phi+\rho S\underline{\nabla}\eta)+(v_{ij}\eta_{,j}+v_{ij,j}\eta)$$
$$+\frac{\Delta t}{\omega}(v_{ij}(\eta_{,t})_{,j}+v_{ij,j}\eta_{,t})\cdot\underline{\nabla}\delta\phi$$
$$+\Big(-\frac{\Delta t}{\omega}(\rho\underline{u}\cdot\underline{\nabla}S_{,t})-\rho\underline{u}\cdot\underline{\nabla}S+\frac{\Phi}{T}\Big)\rho\underline{u}\cdot\underline{\nabla}\delta\eta$$
$$+\Big(-\frac{\Delta t}{\omega}(\rho\underline{u}\cdot\underline{\nabla}\eta_{,t})-\rho\underline{u}\cdot\underline{\nabla}\eta-\frac{p}{R}\Big)\rho\underline{u}\cdot\underline{\nabla}\delta S\Big]d\Omega$$
$$+\int_{\Gamma_\phi}\rho\underline{u}\cdot\underline{n}\,\delta\phi\, d\Gamma=0 \qquad (20)$$

where

$$v_{ij}=\frac{4\mu}{T}\Big(e_{ij}-\frac{1}{3}e_{kk}\delta_{ij}\Big)$$

The resulting equation will be used as the equation for finite element approximation. The dependent variables ϕ, η, S, and test functions $\delta\phi$, $\delta\eta$, δS are interpolated by an isoparametric shape function $\underline{N}^T$

$$A^e(x,y,z,t)=\underline{N}^T(\underline{x})\,\underline{A}^e(t) \qquad (21)$$

where A stands for ϕ, η, S, $\delta\phi$, $\delta\eta$, and δS.

By subtituting equation (21) into (20), one can obtain finite element equations

$$\underline{A}^n\dot{\underline{X}}^n=\underline{R}^n \qquad (22)$$

where

$$\dot{\underline{X}}^n=\begin{bmatrix}\dot{\underline{\phi}}^n\\ \dot{\underline{\eta}}^n\\ \dot{\underline{S}}^n\end{bmatrix}=\frac{1}{\omega}\begin{bmatrix}\underline{\phi}^{n+1}-\underline{\phi}^n\\ \underline{\eta}^{n+1}-\underline{\eta}^n\\ \underline{S}^{n+1}-\underline{S}^n\end{bmatrix}$$

$$\underline{A}^n=\begin{bmatrix}\underline{K}^n_{\phi\phi} & \underline{K}^n_{\phi\eta}-\underline{K}^n_{vi} & \underline{K}^n_{\phi s}\\ \underline{0} & \underline{K}^n_{s\eta}+\underline{K}^n_{da} & \underline{0}\\ \underline{0} & \underline{0} & \underline{K}^n_{s\eta}+\underline{K}^n_{da}\end{bmatrix}$$

$$\underline{R}^n=\begin{bmatrix}\underline{f}^n_\phi-\underline{K}^n_{\phi\phi}\,\underline{\phi}^n-(\underline{K}^n_{\phi\eta}-\underline{K}^n_{vi})\,\underline{\eta}^n\\ \underline{G}^n_s-\underline{K}^n_{s\eta}\,\underline{\eta}^n\\ \underline{G}^n_{vi}-\underline{K}^n_{s\eta}\,\underline{S}^n\end{bmatrix}$$

$$\underline{K}_{\phi\phi}=\sum_e\int_{\Omega^e}\rho\,\underline{N}_{,i}\,\underline{N}^T_{,i}\,d\Omega$$

$$\underline{K}_{\phi\eta} = \sum_e \int_{\Omega^e} \rho S \underline{N}_{,i} \underline{N}^T_{,i} \, d\Omega$$

$$\underline{K}_{\phi s} = \sum_e \int_{\Omega^e} \rho \eta_{,i} \underline{N}_{,i} \underline{N}^T_{,i} \, d\Omega$$

$$\underline{K}_{s\eta} = \sum_e \int_{\Omega^e} \rho u_i \underline{N}_{,i} (\rho u_j \underline{N}_{,j})^T \, d\Omega$$

$$\underline{G}_{vi} = \sum_e \int_{\Omega^e} \frac{\Phi}{T} (\rho u_i \underline{N}_{,i}) \, d\Omega$$

$$\underline{K}_{vi} = \sum_e \int_{\Omega^e} v_{ij} \underline{N}_{,i} \underline{N}^T_{,j} + v_{ij,j} \underline{N}_{,i} \underline{N}^T \, d\Omega$$

$$\underline{G}_s = \sum_e \int_{\Omega^e} -\frac{p}{R} (\rho u_i \underline{N}_{,i}) \, d\Omega$$

$$\underline{f}_\phi = \sum_e \int_{\Gamma_\phi} (\rho \underline{u} \cdot \underline{n}) \underline{N} \, d\Gamma$$

$$\underline{K}_{da} = (\text{ damping factor }) \times \int_{\Omega^e} \underline{N}_{,i} \underline{N}^T_{,i} \, d\Omega$$

$\underline{K}_{da}$ is used to prevent the appearance of numerical disturbances produced by convective operator, and the small values of $\underline{K}_{s\eta}$, $\underline{K}_{vi}$ near the stagnation region. The addition of this damping does not change the solutions when steady state is reached.

By examining equation (22), one can observe that the solutions can be obtained in an uncoupled sequence by calculating

$$\dot{\underline{\eta}}^n = (\underline{K}^n_{s\eta} + \underline{K}^n_{da})^{-1} (\underline{G}^n_s - \underline{K}^n_{s\eta} \underline{\eta}^n) \tag{23}$$

$$\dot{\underline{S}}^n = (\underline{K}^n_{s\eta} + \underline{K}^n_{da})^{-1} (\underline{G}^n_{vi} - \underline{K}^n_{s\eta} \underline{S}^n) \tag{24}$$

first by frontal method[27]. The solution of $\dot{\underline{\phi}}^n$ is then calculated by substituting (23), (24) into the first equation of (22). The solution is advanced from time $n\,\Delta t$ to $(n+1)\Delta t$ until the steady state solution is reached. The detailed solution procedures can be found in references [17,18].

Numerical results

The accuracy of the developed three–dimensional code is evaluated by comparing the available theoretical results of two–dimensional, steady, incompressible laminar flows between two parallel plates. The test problem is designed to analyze low Mach number flows over the geometric configuration of high aspect ratio in z–direction (Fig.1), $\Delta x : \Delta y : \Delta z = 2 : 1 : 10$, since the theoretical results require two–dimensional incompressible flow over $(x - y)$ plane.

The velocity vector plot over half of the developing channel in length 1.92 m is shown in (Fig.2) where $u = 1$ m/sec, $p = 122$ N/m^2, $M = 4.528\times10^{-3}$, $\rho = 3.5\times10^{-3}$kg/m^3, $T = 121.5^{o}$K, $Re = 153$ at the inlet. The computed x–component velocities in (Fig.3,4,5) at locations $x = 0.115$m, 0.46m, 1.035m respectively are in agreement with those by P. A. Longwell[28] and H. Schlichting[14]. The comparsion of fully developed velocity profile with the theoretical first–order result is illustrated in (Fig.6). The contours of velocity in x direction and vorticity are shown in (Fig.7,8) respectively, where the length scale in y direction is enlarged by seven time for easy observation.

The velocity vector plot of inlet Reynold number 2750 flow is shown in (Fig.9) where $P_{in} = 2193$ N/m^2, $\rho = 6.3\times10^{-3}$ kg/m^3, $T = 121.5^{o}$K. The solution of higher Reynold number flow, Re = 25000 for example, can also be obtained without difficulty. The velocity plot over the divergent channel with a 5 degree inclination in illustrated is (Fig.10) where the inlet Reynold number Re is 975.

Conclusion

A set of equations for describing Navier–Stokes flows from the present variational principle is equivalent to those in primitive variables form. It provides a direct extension of Bateman's Principle for inviscid flows to viscous counterparts. The representation of inviscid potential flows in terms of velocity potential can be considered as an alternate of Clebsch transformation. One can easily make comparsions of potential, Euler and Navier–Stokes solutions by using the same numerical procedure. Such an approach provides computational efficiency for the solution of Navier–Stokes flow by starting with the solution of inviscid flow. Considerable computing time may be reduced if different levels of flow simplification are made in the case of multi–dimensional flow problems.

The variational principle for laminar Navier–Stokes flows is further verified numerically by finite element approximation. The computed results of developing entrance channel flows with different Reynold numbers are in agreement with the available theoretical and finite difference solutions. The block–structured numerical scheme for complex flows will be included for adapting the advantages of present unified formulation.

Nomenclature

E : internal energy per unit mass

e_{ij} : rate of strain tensor

f	:	normal mass flux
g	:	normal entropy
H	:	stagnation enthalpy
H_o	:	reference constant for internal energy of a particle
$\underline{K}$	:	coefficient matrix of finite element equations
n_i	:	components of normal vector
p	:	static pressure
R	:	gas constant
S	:	entropy
T	:	temperature
Γ	:	boundary domain
γ	:	ratio of specific heats
δ	:	variational operator
η	:	Lagrangian multiplier
μ	:	laminar viscosity
Π	:	variational functional
ρ	:	mass density
Φ	:	viscous dissipation function
u_i	:	velocity component in i direction
k_{ij}	:	Lagrangian multiplier
h	:	enthalpy
$\underline{\omega}$	:	vorticity vector
$\frac{D}{Dt}$	:	total derivative
$,_j$	:	derivative with respect to j
δ_{ij}	:	Kronecker's delta function
ω	:	relaxation factor
$\underline{N}$	:	shape function
Re	:	Reynold's number
ϕ	:	Lagrangian multiplier
S	:	entropy

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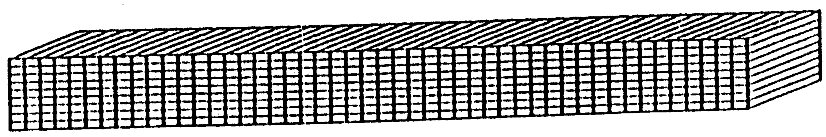

Figure 1 Computational Grid for the Flow within a Straight Duct (48 × 9 × 1 Elements, 980 Nodal Points)

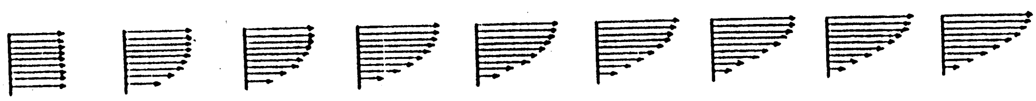

Figure 2 Velocity Vectors at Different Sections of a Straight Duct (Inlet Re= 153, Δx= 0.24 m)

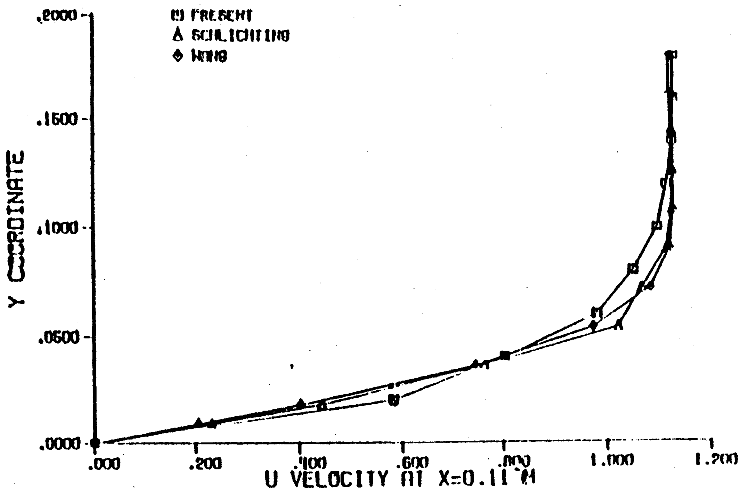

Figure 3 Comparisons of U Velocity with W.L. Wang and Schlichting at Location x= 0.11 m.

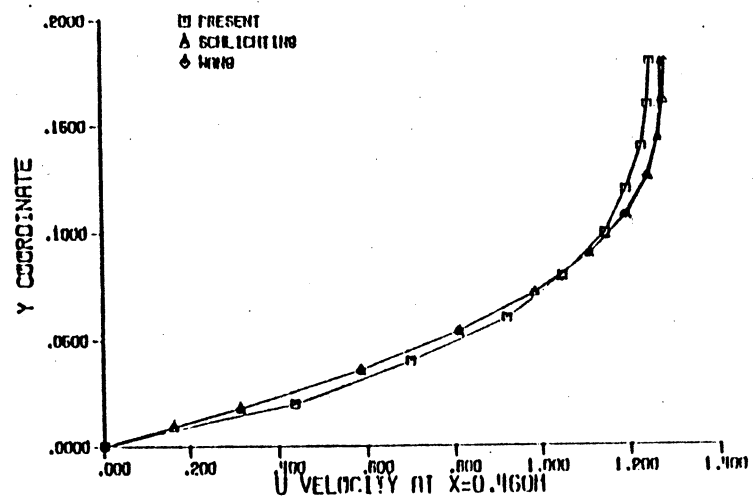

Figure 4 Comparisons of U Velocity with W.L. Wang and Schlichting at Location x= 0.46m.

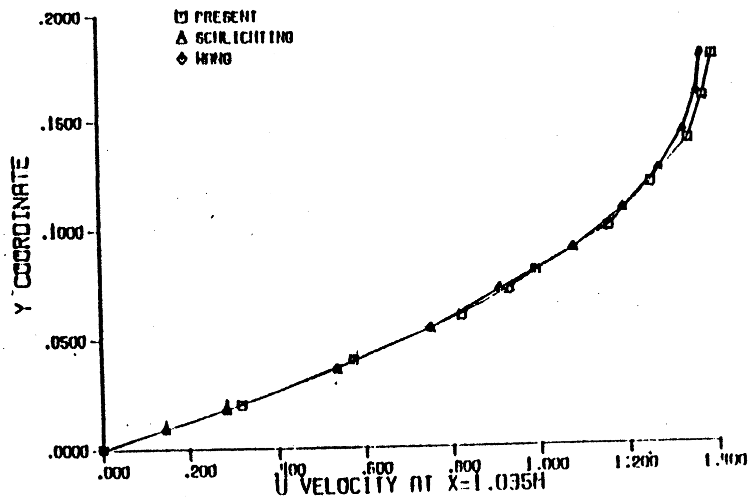

Figure 5 Comparisons of U Velocity with W.L. Wang and Schlichting at Location x= 1.035m.

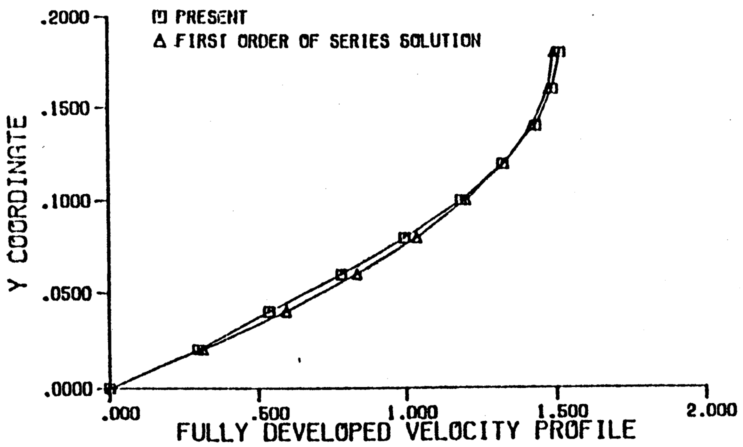

Figure 6. Comparisons with the Theoretical Results (uniform Inlet Re= 153).

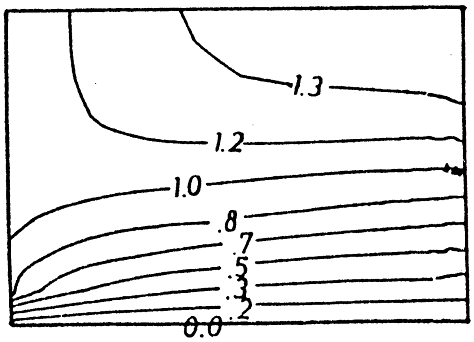

Figure 7 U Velocity Contours for the Flow within a Straight Duct (Re= 153)

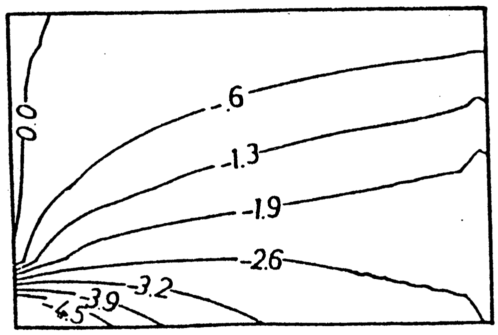

Figure 8 Vorticity ω Contours for the Flow within a Straight Duct (Inlet Re= 153)

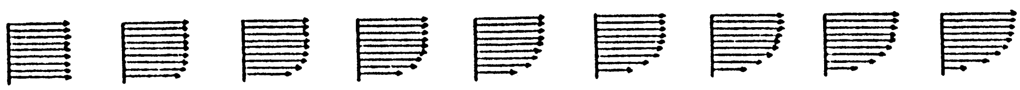

Figure 9 Velocity Vectors at Different Sections of a Straight Duct (Uniform Inlet Re= 2750, Δx= 0.24 m)

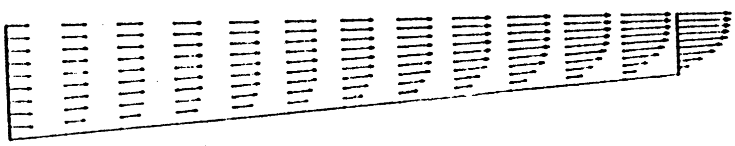

Figure 10 Velocity Vectors for the Flow within a Convergent Duct (Inlet Re= 975)