A MONOTONE MULTIDIMENSIONAL UPWIND FINITE ELEMENT METHOD FOR ADVECTION-DIFFUSION PROBLEMS

Tony W. H. Sheu, S. F. Tsai, and Morten M. T. Wang
Department of Naval Architecture and Ocean Engineering, National Taiwan University, 73 Chou-Shan Rd., Taipei, Taiwan, Republic of China

We are interested in developing a multidimensional convective scheme that is capable of dealing with erroneous oscillations near jumps. The strategy is based on the Petrov-Galerkin formulation, to which the underlying idea of the M matrix is added. The nature of the exponentially weighted upwind method is best illuminated by its matrix structure. We interpret the enhanced stability as being due to the attainable irreducible diagonal dominance. The accessible monotonicity condition enables us to construct a monotone stiffness matrix a priori, thereby laying the foundation for arriving at the monotonicity-preserving property. In order to show the merit of the proposed upwinding technique in resolving spurious oscillations generated by unresolved internal and boundary layers, we considered two classes of convection-diffusion problems. As seen from the computed results, we can attain an accurate finite-element solution for a problem free of boundary layer and can capture a high-gradient solution in the sharp layer.

INTRODUCTION

Numerical prediction of transport equations is an area of continuing progress because of its practical relevance. With the extensive numerical experiments of the past few decades and the advent of high-speed computers, approximations to partial differential equations of this sort have reached a high degree of maturity. Nevertheless, much work is still being done to circumvent notorious difficulties in association with flux discretizations in multiple dimensions. It is fair to say that simulation quality is influenced largely by the convective term (first derivative). Among the properties worthy of being pursued, the attainment of one property may violate the other. For instance, use of an upwind scheme provides a stabilized means for solving problems involving high values of the Peclet number. This, however, leads to deterioration of simulation quality in that excessive artificial viscosities overspread the solution profile and thereby contaminate the real physics. Accordingly, researchers have strong motivation to build a sound basis for resolving this dilemma.

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Address correspondence to Dr. Tony W. H. Sheu, Department of Naval Architecture and Ocean Engineering, National Taiwan University, 73 Chou-Shan Road, Taipei, Taiwan, Republic of China.

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NOMENCLATURE

B_i (i = 1–4)	biased part of weighting functions	u, v	velocity components in x and y directions
c_i, d_i, e_i	coefficients in Eq. (7)	W_i (i = 1–4)	weighting functions
C_i (i = 1–3)	coefficients in Eq. (10)	x, y	Cartesian coordinates along physical plane
D	physical domain	Γ	boundary of D
h	mesh size in the computational plane	θ	angle of the flow direction
N_i (i = 1–4)	shape functions	μ	coefficient of diffusivity
P_e	Peclet number	ξ, η	computational coordinates in the transformed plane
T	truncation error in the convection-diffusion equation	Φ	working variable

The finite-element method has enjoyed success in solid mechanics and heat conduction for years. Applied to the field of fluid dynamics in the middle 1960s, finite elements have a rich mathematical background from which one can prove convergence. Like the recent developments in the use of body-fitted coordinates with finite-volume discretization techniques, the finite-element method has an appealing attraction in handling geometric complexity. Also, differential equations involving the Neumann-type boundary condition are more amenable to the finite-element method. These advantages provide a strong impetus for utilization of the Galerkin-type methodology presented here. The reasons why this methodology finds preference in simulating flows characterized by high Peclet numbers for the convection-diffusion equation will be given later.

Classical Galerkin finite-element methods, being the equivalent centered finite-element methods, have often yielded unphysical oscillatory solutions in situations where convective terms significantly dominate diffusive terms. Physical reasoning suggests that a nonsymmetric treatment of convective terms would be more appropriate. While this approach is of great aid in stabilizing the calculations, it simultaneously brings in unwanted cross-stream diffusion. This artifact smears the solution in areas of high gradients present in the flow field. The cure for such pathologic oscillations is to modify the weighting functions underlying the streamline upwind Petrov-Galerkin (SUPG) framework [1] so as to secure biased functions in favor of the upstream side.

Another method falling into the category of the streamline upwind method is that of Rice and Schnipke [2]. Its success in yielding streamline damping is attributable to the evaluation of the convection term along the local streamline. Besides this class of methods, numerous concepts have appeared in the literature for gaining access to higher-order accuracy while retaining numerical stability. These cures hold easily in one dimension. For a fairly small value of diffusivity, the problem considered tends to be hyperbolic because the reduced problem (zero diffusivity) is essentially hyperbolic. In circumstances where time-dependent pure advection equations are considered, monotonicity-preservation solutions are available in the one-dimensional context. One quickly learns through practice that high-resolution solutions are not easily obtained in multiple dimensions, mostly due to the lack of a viable stability criterion to follow. In order to design an effective multidimensional advective flux scheme, we have felt the need for a treatise on this issue. Mizukami and Hughes [3] extended the application range of SUPG to flow problems containing a sharp layer by demanding that the discrete system underlying the streamline upwind scheme ensure the satisfaction of the maximum principle [4–6]. It is this maximum principle that makes the monotonic behavior of the solution variable accessible. Hill and Baskharone [7] took an alternative route to reach a monotone solution. The idea behind their success in yielding the so-called monotonicity-preserving property was to approximate the weighted residuals of convection terms in steamline coordinates. On each element, the advective flux is preserved along the streamline. Evaluation of these fluxes requires definition of an upwind point a priori. In light of the work of Ahue and Telies [4], we can select a legitimate biased weighting so as to yield a monotone matrix. Oscillations present in the narrow region of high gradients thus can be well suppressed. Of note is that this method is intended for modeling discontinuities in a domain of multiple dimensions.

We begin by describing the target problem, known as the convection-diffusion equation. We explain in detail why the proposed Petrov-Galerkin method possesses the monotonicity-preserving property. Since both solution accuracy and stability have great influence on simulation quality, we have also conducted fundamental studies regarding the stability and accuracy aspects. In order to validate the proposed flux discretization scheme, we present three closed-form solutions for the scalar transport equation defined in a square cavity. Attention is directed to assessing the performance of the scheme.

MULTIDIMENSIONAL FLUX DISCRETIZATION SCHEME

Model Equation and Discretization Method

Over the past few decades, numerical simulation of a transport field variable in two dimensions has been the subject of many intensive studies in the CFD community. While numerical analysis of this class of differential equations is important in itself, its real focus lies in its resemblance to the linearized equations of motion for incompressible fluid flows, or of the electron and hole continuity equations in semiconductor device modeling. This topic has been, historically, of interest to the aerospace and processing industries. They have placed emphasis on fluids and flow speeds with high Peclet or Reynolds numbers. As a model problem, we consider the convection-diffusion equation in a homogeneous medium. We analyze scalar transport equation for simplicity, but never embark on a course that would preclude system equation generalization. In a simply connected domain D, we restrict our attention to a simpler case involving a constant diffusion coefficient and velocities. Simply stated, the solution to the following elliptic system is sought:

$$\begin{gathered} u\Phi_x + v\Phi_y = \mu(\Phi_{xx} + \Phi_{yy}) \qquad \text{in } D \\ \Phi = g \text{ on } \Gamma = \partial D \end{gathered} \tag{1}$$

This problem can be viewed as modeling of a steady-state convection-diffusion equation, in which Φ represents some transported quantities. Here, (u, v) is the advection velocity vector, and μ is the coefficient of diffusivity. Without loss of generality, we assume that u, v, and μ are positive constants. According to the prescribed boundary data, g, interior solutions are the result of convection and diffusion effects. The relative importance is best measured by the maximum Peclet number,

$$(P_{ex}, P_{ey}) = \max\left(\frac{u\,\Delta x}{\mu}, \frac{v\,\Delta y}{\mu}\right)$$

where Δx and Δy denote mesh sizes along the x and y directions, respectively. In this elliptic context, we are interested in flow conditions having dominant first derivatives.

The finite-element method has been used extensively to solve fluid flow problems. The Galerkin-based, collocation, and least-squares methods are well-known weighted residuals variants [8]. Among them, a formulation may be regarded as advantageous from the prediction accuracy viewpoint rather than from the standpoint of stability or monotonicity. In situations where convective terms are overwhelmingly dominant over diffusive terms, analysis of this class of flows necessitates the use of upwinding procedures. To this end, we proceed along the line of the method of weighted residuals by demanding that the residual $R = u\Phi_x + v\Phi_y - \mu(\Phi_{xx} + \Phi_{yy})$ be orthogonal to the weighting function. The solution sought, then, can be seen as a search for the weak solution to Eq. (1). By substituting finite-element approximation for the working variable Φ, we are led to an element-based matrix equation:

$$\left[\int_{A^e} W_i\left(u\,\frac{\partial N_j}{\partial x} + v\,\frac{\partial N_i}{\partial y}\right) dA\right]\Phi_j + \left(\mu\int_{A^e} \frac{\partial W_i}{\partial x}\,\frac{\partial N_j}{\partial x} + \frac{\partial W_i}{\partial y}\,\frac{\partial N_j}{\partial y}\,dA\right)\Phi_j = 0 \quad (2)$$

where A^e is the element area. Upon assemblage of all elements, the global coefficient matrix is formed. What remains is determination of the weighting function, which is of pivotal importance and is a rather obscure issue, before the weak solution can be computed from a direct solution solver.

Construction of Test Space

As progress has been made in the area of flux discretization for the advection-diffusion equation, continued research attention has been directed toward pursuing conservativeness, convective stability, and boundedness properties. Usually, upwind finite-element models accommodate the first two desirable properties. Whether or not an upwind scheme can render the boundedness property is closely related to the sign and absolute values of the components in the resulting finite-element coefficient matrix. Diagonal dominance serves as a sufficient condition to assure bounded solutions. If this condition holds, nonphysical overshoots or undershoots are avoided. In light of this, we resort to using the Petrov-Galerkin formulation. The question of which weighting functions to adopt so as to yield a monotone solution remains open. We elaborately construct a coefficient matrix which is of some importance, for it amounts to determining whether the finite-element solution to the scalar transport equation is sensitive to sharp solution gradients. Being motivated by the work of Ahue and Telies [4], we will construct a new class of test functions. The underlying idea evolves from the use of weighting functions falling into the exponential setting. As is well known, thus far, one is prone to make the matrix equation to satisfy the monotonic condition a priori, thereby leading to an **M** matrix, when adopting a biased model.

It has been well known that employment of total variation diminishing (TVD) conditions enables us to have a high-resolution solution for the one-dimensional hyperbolic equation. Inspired by this fact, we wish to find out whether there exists a general rule to follow so that a monotonic solution is also amenable to an elliptic problem defined in a multidimensional domain. Before turning to construction of such a test space, it is convenient to present useful theorems [4–6]. Also, some relevant definitions are summarized as follows for the sake of description [4–6].

Definition 1: A real $n \times n$ matrix $\mathbf{A} = (a_{ij})$ is said to be irreducible diagonally dominant if $|a_{ii}| > \sum^{n}_{\substack{j=1 \\ j \neq i}} |a_{ij}|$ for at least one i.

Theorem 1: If $\mathbf{A} = (a_{ij})$ is a real, irreducible, diagonally dominant $n \times n$ matrix with $a_{ij} \leq 0$ for $i \neq j$ and $a_{ii} > 0$ for $1 \leq i \leq n$, then $\mathbf{A}^{-1} > 0$.

Definition 2: A real $n \times n$ matrix $\mathbf{A} = (a_{ij})$ with $a_{ij} \leq 0$ for all $i \neq j$ is an **M** matrix if **A** is nonsingular and $\mathbf{A}^{-1} > 0$.

Definition 3: A real $n \times n$ matrix **A** is defined as monotone if $\mathbf{A}\boldsymbol{\phi} \geq 0$ holds for any vector $\boldsymbol{\phi}$, it implies $\boldsymbol{\phi} \geq 0$.

Theorem 2: If the off-diagonal entries of **A** are nonnegative, we are led to a monotone **A** if and only if **A** is an **M** matrix.

As the core of the present analysis, these definitions and theorems provide a stabilized means for forming a well-conditioned matrix equation. Inclusion of the idea of an **M** matrix is intended to cope with oscillations about jumps. In the finite-element method using the weighted-residuals formulation, we use bilinear quadrilateral element shape functions:

$$N_i(\xi, \eta) = \tfrac{1}{4}(1 + \xi_i \xi)(1 + \eta_i \eta) \tag{3}$$

The test space is constructed by modifying the shape functions through the use of a biased exponential polynomial:

$$W_i = N_i + B_i \tag{4}$$

where

$$B_i(\xi, \eta) = \left\{ \exp\left[-\frac{u h_x}{2\mu}(\xi - \xi_i) \right] \exp\left[-\frac{v h_y}{2\mu}(\eta - \eta_i) \right] - 1 \right\} N_i(\xi, \eta) \tag{5}$$

In Eq. (2), h_x and h_y denote grid sizes.

Figure 1 illustrates the piecewise weighting function in regions containing the node (i, j), which are surrounded by four bilinear elements. This figure clearly illuminates why the discrete system has been enhanced by the exponential function defined in Eqs. (4)–(5). Here, we consider only the scalar problem in two dimensions. For extension of this idea to three dimensions or to a system of equations, one can follow the same procedure.

Fundamental Study of Accuracy

To gain insight into the behavior of the convection-diffusion scheme, we have conducted a fundamental study of the employed exponential upwind method as applied to a steady advection-diffusion model problem. Of numerical properties, the stability and accuracy deserve detailed discussion. In an attempt to obtain knowledge of discretization errors, we derive the modified partial differential equation of Warming and Hyett [9]. The task of deriving the accompanying discretization error T, shown as follows, demands quite tedious manipulations:

$$\mathbf{u} \cdot \nabla\Phi - \mu\Delta\Phi = T \tag{6}$$

where

$$T = c_1\Phi_{xx} + c_2\Phi_{xy} + c_3\Phi_{yy} + d_1\Phi_{xxx} + d_2\Phi_{yyy} + d_3\Phi_{xxy} + d_4\Phi_{xyy} + e_1\Phi_{xxxx} + e_2\Phi_{xxxy} + e_3\Phi_{xxyy} + e_4\Phi_{xyyy} + e_5\Phi_{yyyy} + \cdots \tag{7}$$

The use of exponential polynomials precludes concise representation of the coefficients in Eq. (7) because algebraic manipulations are considerable. We thus have plotted only the leading coefficients logarithmically against the grid sizes. Figure 2 reveals that the rates of convergence for Φ_{xxxy} and Φ_{xyyy} are $O(h^4)$, while those for Φ_{xy}, Φ_{xxx}, Φ_{yyy}, Φ_{xxy}, Φ_{xyy}, Φ_{xxxx}, Φ_{yyyy}, and Φ_{xxyy} are in the vicinity of

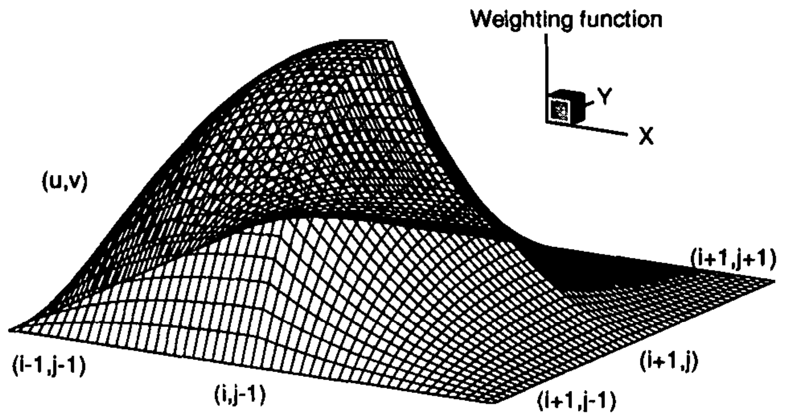


Figure 1. Illustration of the weighting function, defined in Eqs. (2)–(4), in a block of four bilinear elements having a common corner node (i, j).

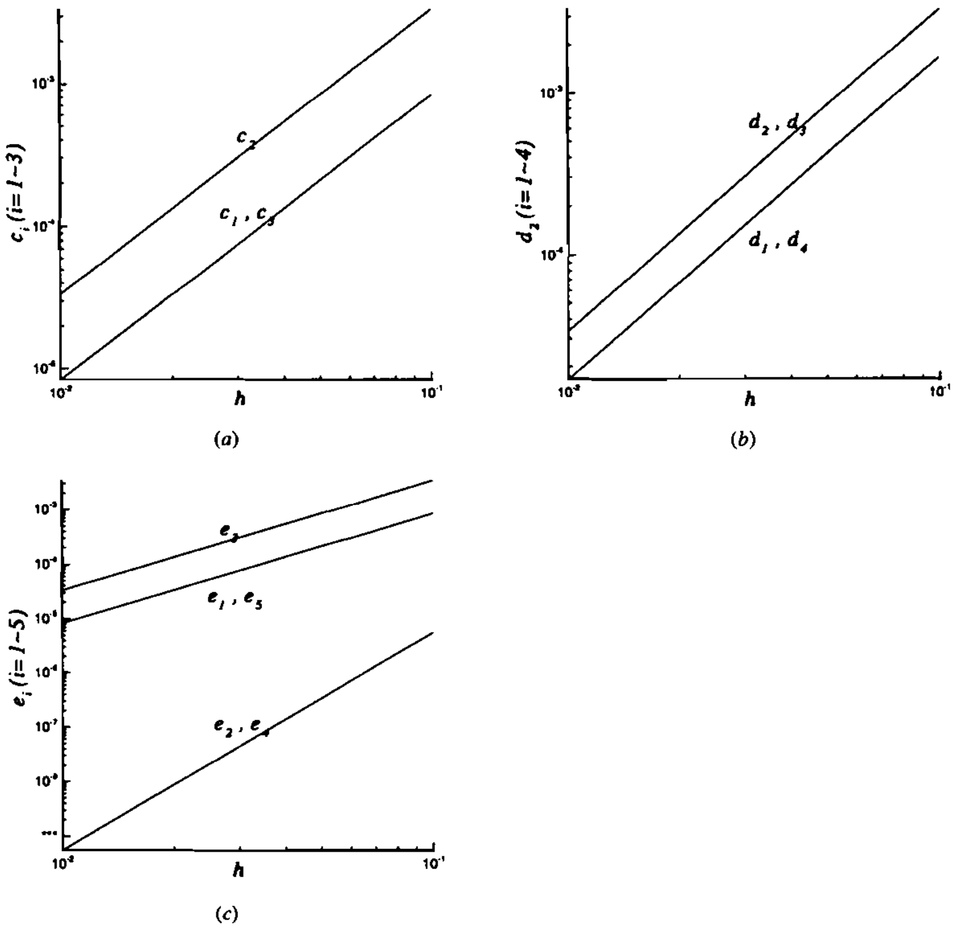


Figure 2. The computed rates of coefficients appearing in Eq. (7): (a) Φ_{xx}, Φ_{xy}, Φ_{yy}; (b) Φ_{xxx}, Φ_{xxy}, Φ_{xyy}, Φ_{yyy}; (c) Φ_{xxxx}, Φ_{xxxy}, Φ_{xxyy}, Φ_{xyyy}, Φ_{yyyy}.

$O(h^2)$. As to the leading error terms, the rates of convergence for Φ_{xx} and Φ_{yy} are exactly two.

With the local error estimates achieved, we will explore in depth the overall order of accuracy for the scheme proposed. To this end, we take the following model problem into consideration:

$$(x + \tfrac{1}{2})^{-1}\Phi_x + (y + \tfrac{1}{2})^{-1}\Phi_y - \nabla^2\Phi = 0 \tag{8}$$

In a square domain of unit length, the analytic solution takes the following form:

$$\Phi = (x + \tfrac{1}{2})^2 + (y + \tfrac{1}{2})^2 \tag{9}$$

By substituting the analytic solution Φ into Eq. (7), we can have the estimate of the global error of T, measured in the L_2-norms sense:

$$\mathrm{err}(\Phi) = \frac{|\Phi - \overline{\Phi}|_2}{|\Phi|_2}$$

where

$$|\Phi - \overline{\Phi}|_2 = \left[\sum_{ij}\left(\Phi_{ij} - \overline{\Phi}_{ij}\right)^2\right]^{1/2}$$

$$|\Phi|_2 = \left(\sum_{ij}\overline{\Phi}_{ij}^2\right)^{1/2}$$

In this article, we also present a rate of convergence test for the sake of completeness. The intention here is to gain insight into the behavior of error reduction as the mesh size decreases. A continuous and uniform grading of meshes is thus required along each dimension. Solutions to the problem under consideration were computed from several uniform grids starting from 5×5. By doubling the number of nodal points in each dimension, we benchmark the scheme performance in terms of the rate of convergence defined below:

$$\text{rate} = \frac{\log(\mathrm{err}_1/\mathrm{err}_2)}{\log(N_1/N_2)}$$

Table 1 reveals errors err_1 and err_2, which were obtained from the grid system having $(N_1 + 1)^2$ and $(N_2 + 1)^2$ points, respectively. With these values, we can estimate the resulting rate of convergence.

As far as the transport equation in multiple dimensions is concerned, it is of importance to know how discretization errors vary, either along or normal to the flow direction $\theta = \tan^{-1}(v/u)$. To accomplish this task, we further transform the truncation error term T from the physical coordinates (x, y) into the streamline

Table 1. Rate of convergence for the test problem defined in Eqs. (7)–(8) based on the proposed discretized advection-diffusion scheme

Element	L_2 norm	Convergence rate
5×5	3.2614×10^{-3}	
		3.36
10×10	3.1738×10^{-4}	
		3.19
20×20	3.4681×10^{-5}	
		3.29
40×40	3.5297×10^{-6}	

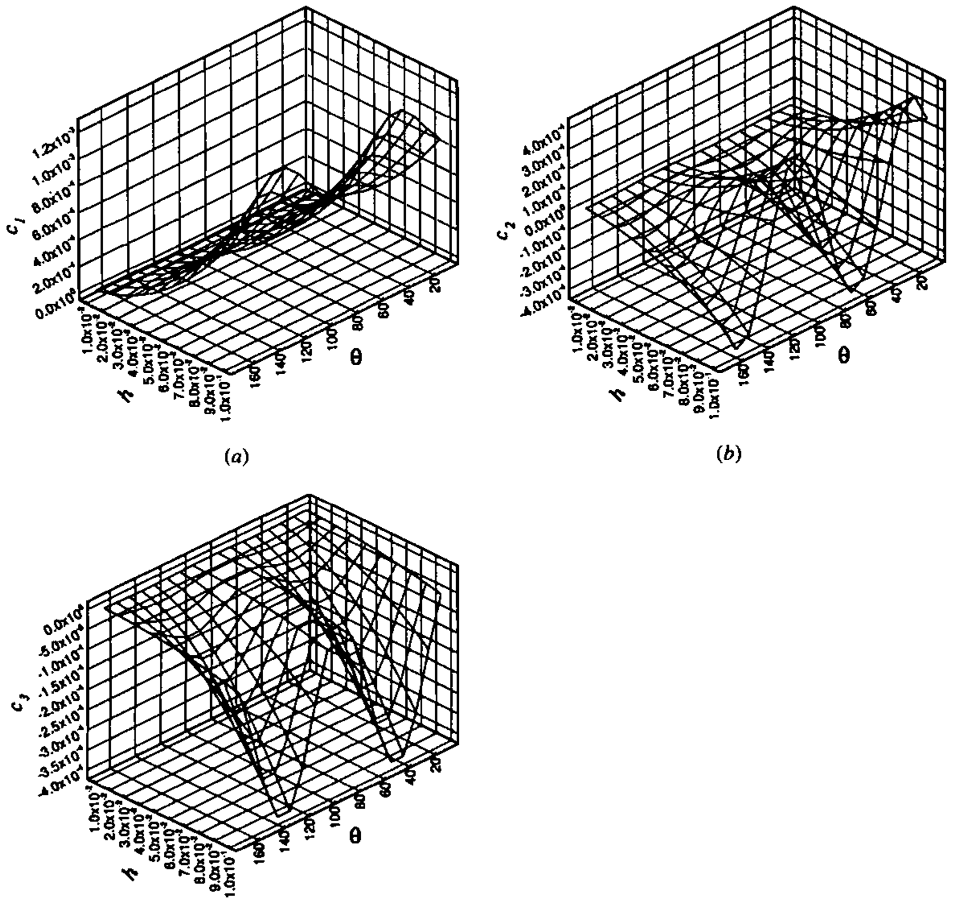


Figure 3. The computed coefficients of C_i ($i = 1 \sim 3$) in Eq. (10) against the grid size h and the flow angle θ: (a) C_1; (b) C_2; (c) C_3.

coordinate s and its normal n through one-to-one mapping. Applying the chain rule yields

$$T = C_1(h,\theta)\Phi_{ss} + C_2(h,\theta)\Phi_{sn} + C_3(h,\theta)\Phi_{nn} + \cdots \tag{10}$$

where h is the uniform grid size.

The efforts to express the coefficients appearing in Eq. (10) in a functional form are also considerable. It is, as a consequence, appropriate to plot them graphically against θ and h. Figure 3a reveals that the values of C_1 are always positive, regardless of the flow angle encountered. This implies that an artificial viscosity of physical plausibility has been added implicitly along the streamline. In contrast to C_1, the computed artificial viscosity along the direction normal to the streamline is negative, as shown in Figure 3b. According to Figure 3, we realize that the numerical stability has been enhanced mainly along the flow direction by implicit addition of the damping term. Flows involving flow angles, namely 45° and 135°, are highly smeared. It is worthwhile to note that when flow analysis is performed in a one-dimensional-like context ($\theta = 0°, 90°$), the truncation error remains only along the streamline direction. Also of note is that if C_2 is locally equal to zero, experience gained from previous numerical exercises indicates that the employed numerical model is by no means unconditionally stable in the presence of a positive value of C_1 and a negative value of C_3. We believe that prediction errors of this sort might be the key to causing the proposed flux discretization scheme to be conditionally monotonic. This prompts us to explore in depth the following stability analysis.

Fundamental Study of Monotonicity

As the flow convection largely dominates diffusion, there is potential loss of ellipticity. Under these circumstances, the solution profile taking the form of a sharp boundary layer may be set up in the interior domain. Very often, classical upwind finite-element methods fail to provide smooth solutions in regions of sharp gradients. In this regard, the selection of a test space that is applicable to the present framework is necessary. We thus represent the discrete finite-element equation in a form analogous to that expressed in the finite-difference setting, namely, $\Sigma_{i=1}^{9} a_i \Phi_i = 0$. The derivation of $a_i = a_i(p_{ex}, P_{ey}, h)$, while algebraically tedious, helps us to decide to what extent the values of (P_{ex}, P_{ey}) still render the irreducible diagonally dominant matrix defined above. For clear demonstration of the so-acquired stable region, we have calculated the coefficients of a_i against P_{ex} and P_{ey}. The shaded area shown in Figure 4 is referred to as the monotonic domain in the sense that finite-element solutions of Eq. (1) are nonoscillatory if the mesh sizes and velocities under consideration correspond to the Peclet number falling into this region. Under these circumstances, a set of nine coefficients follows the definition of the **M** matrix. While we can gain access to a monotonic solution through the approximation underlying the concept of the **M** matrix, this stability criterion may be too restrictive for us to ameliorate the computational accuracy. No consideration will be given here to optimizing this compensating factor. Of note is that when the analysis is reduced to one dimension, the discretization equation proposed here turns out to be identical to that of the localized adjoint method [10]. With the accessible consistency property discussed in the previous section, the convergence solution meets our expectations as long as the problem considered and the mesh discretized cause the values of P_{ex} and P_{ey} to fall into the realm of a stable region.

CHARACTERISTIC REFINEMENT OF THE EXPONENTIAL UPWIND SCHEME

While monotonic finite-element solutions of Eq. (1) are possible provided that Peclet numbers fall into the monotonic region as discussed in the preceding section, the restriction of using Peclet numbers smaller than 3.2 precludes the extension of the proposed model to a wider range of applications. In order to

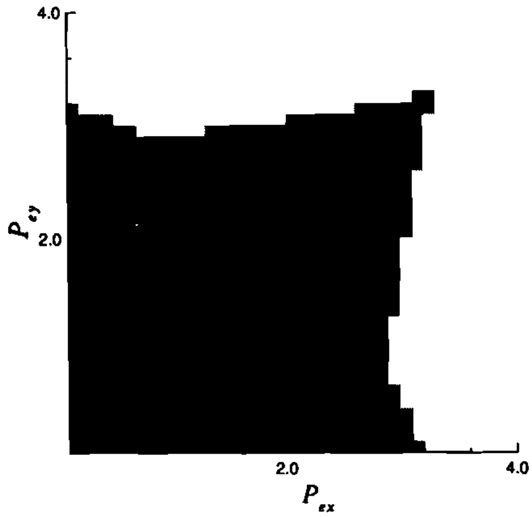


Figure 4. Illustration of the monotonic region underlying the proposed upwind scheme.

remove this constraint, we bring in the concept of the method of characteristics because the higher the Peclet numbers are, the more the differential system tends to be singularly perturbed. For a given interior spatial point, physical reasoning tells us that we do not need to take each stiffness matrix involving this point into consideration in the assemblage procedures. Instead of taking four elements into account, we consider only the element upsteam of this point in mimicking the characteristic behavior. To demonstrate the feasibility of this characteristic extension, we will present finite-element results for benchmark test problems presented in the results section.

NUMERICAL STUDY

To make clear whether or not the proposed Petrov-Galerkin finite-element model has the monotonicity property, we will consider a smooth advection-diffusion problem that is amenable to analytic solution. For completeness, problems with a boundary layer and internal layer are considered in a domain of two dimensions.

Problem without a Boundary Layer

The first example we will deal with is that of a smoothly varying transport problem, as defined in Figure 5. Within the physical boundary, along which prescribed boundary data are given, the velocity field is assumed to be constant, that is, $\mathbf{u} = (1, 10.5)$. This problem has been used by Arampatzis and

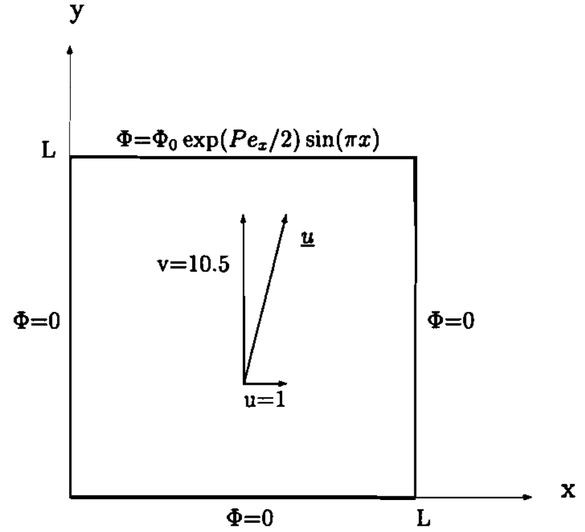


Figure 5. Illustration of the test problem without a boundary layer.

Assimacopoulos [11] to benchmark the discretization scheme. Given these velocities and boundary data, the analytic solution to Eq. (1) takes the following form:

$$\Phi = \frac{\Phi_0}{e^{r_+} - e^{r_-}} e^{P_{ex}/2} \sin(\pi x)(e^{r_+ y} - e^{r_- y})$$

where

$$r_{\pm} = \frac{1}{2} P_{ey} \pm \frac{1}{2} \sqrt{\left(P_{ey}^2 + 4\beta\right)} \qquad \beta = \frac{(4\pi^2 + P_{ex}^2)}{4}$$

In a 15 × 15 uniformly discretized domain, the computed relative error, measured in terms of the so-called L_2 norm, is 2.434×10^{-3}. We have plotted the computed error at $x = 0.5$ against y in Figure 6 for the purpose of comparison with the QUICK solution given in [11]. Both prediction solutions deviate largely from exact solutions in the vicinity of $x = 1$, where the solution has higher gradients, as shown in Figure 7. As y decreases, progressively smaller errors result. As indicated by this figure, overwhelming superiority to the third-order QUICK scheme is demonstrated.

Problem with a Boundary Layer

Encouraged by the close agreement between the calculation and the analytic data, we can make an attempt to deal with a stringent problem that is characterized as possessing a sharp boundary layer. A good approach to demonstrating the integrity of the underlying method to resolve a sharp profile is to solve a problem amenable to the analytic solution. The test problem attempted here is subject to the boundary values specified in Figure 8. The analytic solution to this problem takes the following form:

$$\Phi(x, y) = \frac{\{1 - \exp[(x - 1)(u/(\mu))]\}}{[1 - \exp(-u/\mu)]} \frac{\{1 - \exp[(y - 1)(v/\mu)]\}}{[1 - \exp(-v/\mu)]}$$

Numerical solutions were sought on several regular grids. Given the computed solutions plotted in Figure 9 and the computed values of L_2-norm error tabulated in Table 2, the present finite-element formulation was confirmed as being able to tackle a problem involving a rapid change of solution inside the flow.

Skew Advection-Diffusion Problem

An even harder problem, which is distinguishable from other test problems in the presence an internal layer, will be considered. This test is that of the skewed flow transport problem, which is regarded as a worst-case scenario for any upwinding method [12]. In the square cavity of unit length, the cavity is divided into two subdomains by a tile line, which passes through the left lower corner at (0, 0), having a slope of $m = \tan^{-1}(v/u)$. Throughout the whole domain, the magnitude of the velocity of interest is maintained as $q = \sqrt{u^2 + v^2} = 1$. Subject to the

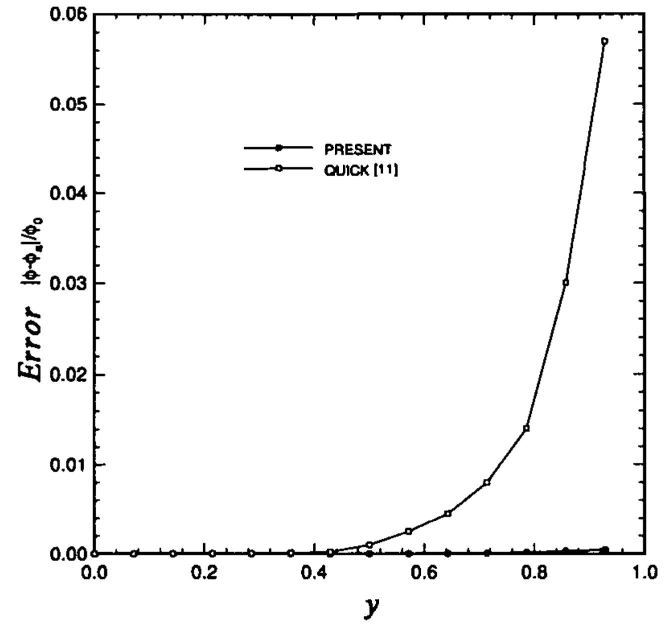


Figure 6. Computed solution profile at $x = 0.5$ for the test problem without a boundary layer.

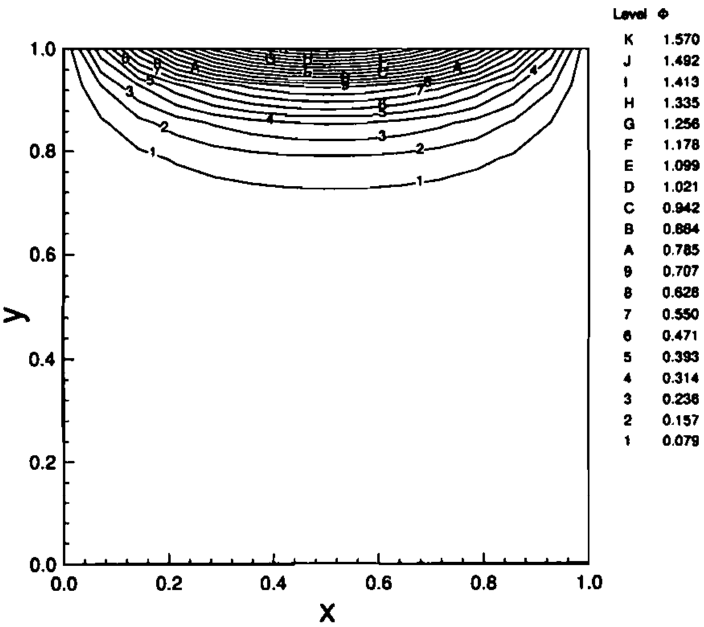


Figure 7. Computed contour plots of Φ for the problem without a boundary layer.

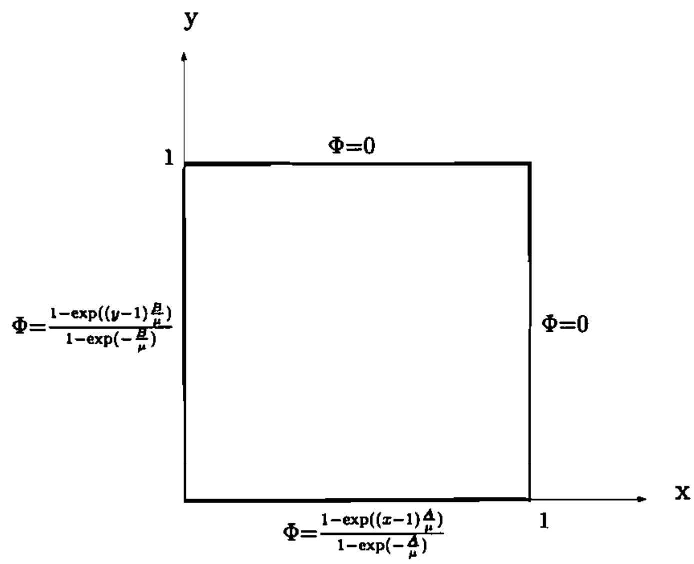


Figure 8. Illustration of the test problem with a boundary layer.

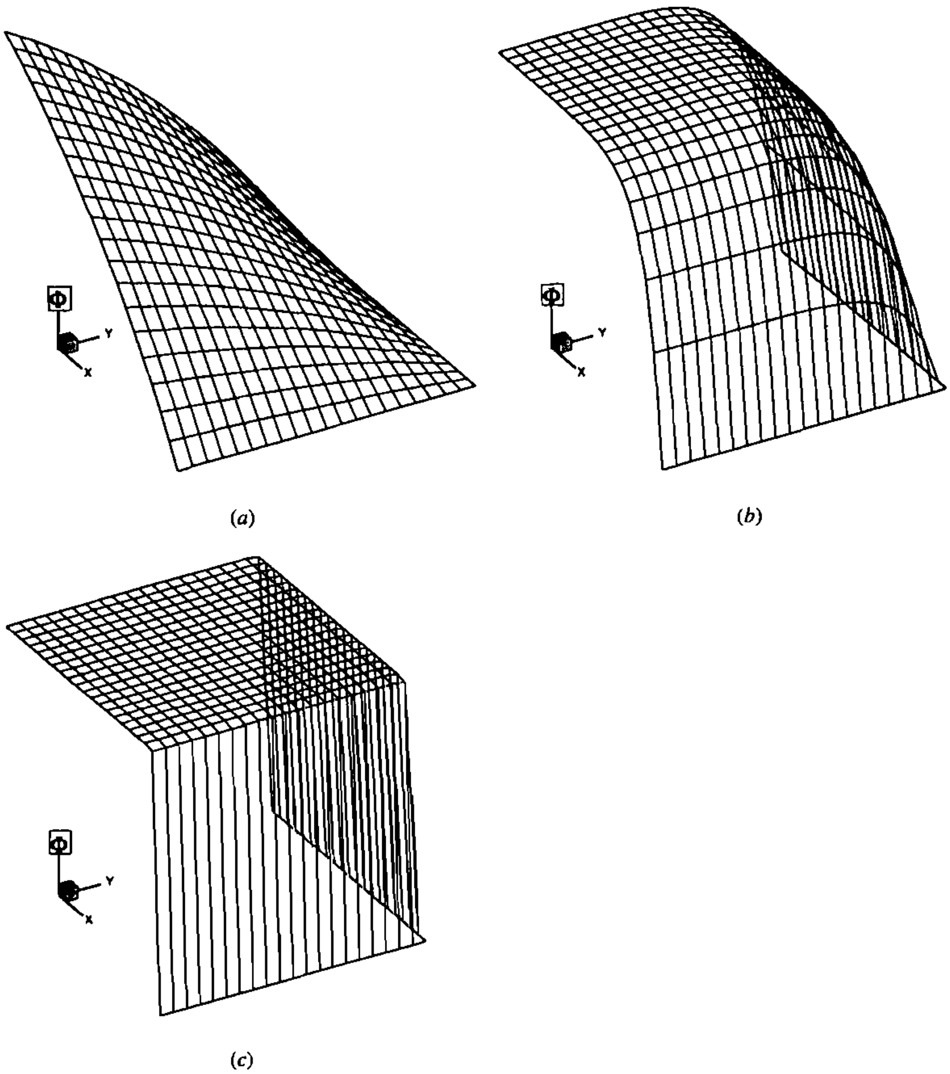


Figure 9. Contour plots of Φ for the test problem with a boundary layer: (*a*) $u = v = 1$; (*b*) $u = v = 10$; (*c*) $u = v = 100$.

Table 2. Computed L_2 norms at different flow conditions for the problem with a boundary layer

Element	$u = v = 1$	$u = v = 10$	$u = v = 100$
10 × 10	1.175×10^{-9}	1.217×10^{-9}	2.902×10^{-6}
20 × 20	9.600×10^{-10}	1.060×10^{-9}	2.781×10^{-8}
40 × 40	6.833×10^{-10}	7.551×10^{-10}	4.958×10^{-10}

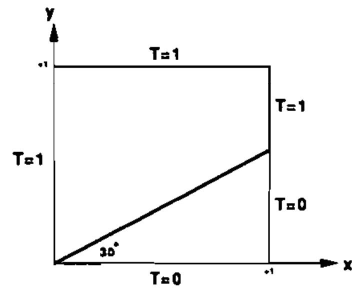


Figure 10. Illustration of the skew advection-diffusion problem.

boundary condition for the working variable given in Figure 10, a shear layer of high gradient or near discontinuity is expected when crossing the dividing line.

The objective of this case is to assess the merit and the deficiency of the proposed upwinding technique. We consider a uniform flow parallel to the dividing line in a 20 × 20 uniformly discretized mesh. Different diffusivities are considered that correspond to different degrees of advection dominance. With the diffusivity set to 1.67×10^{-2}, the cell Peclet number approaches 3, which falls into the monotonic regime. Finite-element solutions shown in Figure 11 are free of oscillations in regions close to as well as away from the dividing line. Monotonic and

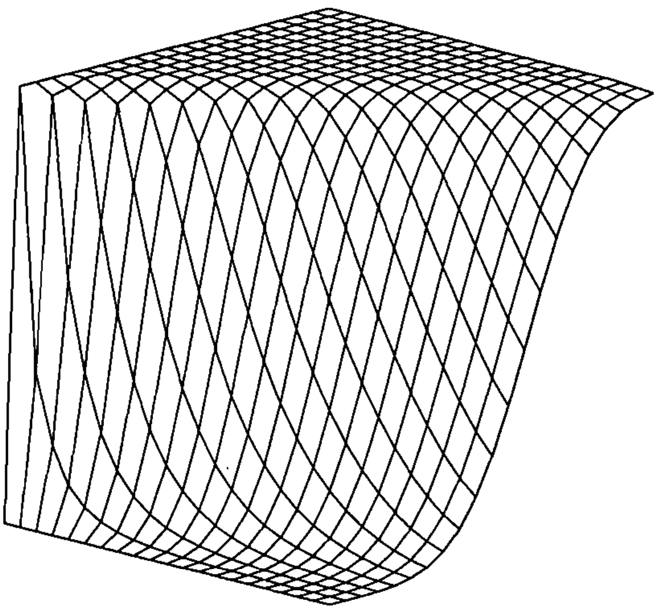


Figure 11. Computed solution of Φ for the case of $\mu = 1.67 \times 10^{-2}$, which corresponds to Peclet numbers falling into the monotonic region.

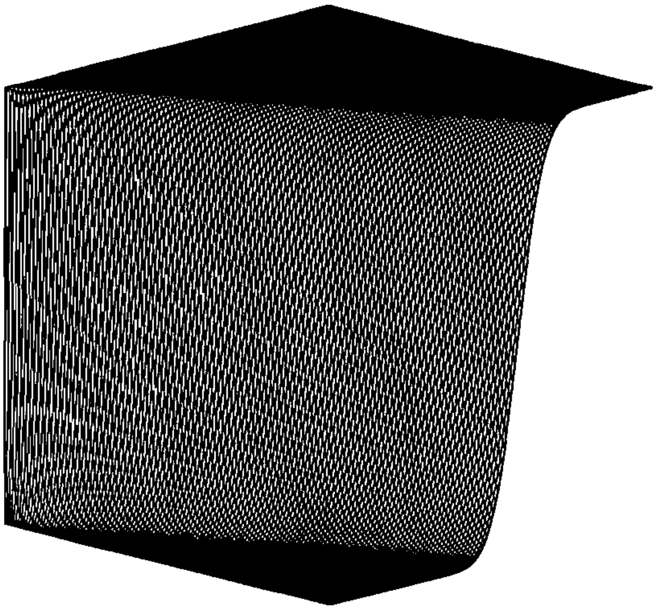
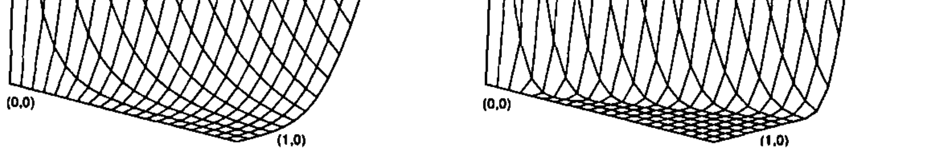


Figure 12. Computed solution of Φ in a 150×150 uniformly discretized domain. The test considered is that of $\mu = 2 \times 10^{-3}$, which corresponds to Peclet numbers falling into the monotonic region.



(*a*) (*b*)

Figure 13. Computed solution of Φ for the case of $\mu = 2 \times 10^{-3}$: (*a*) characteristic version of the proposed finite-element model; (*b*) characteristic-free finite-element model.

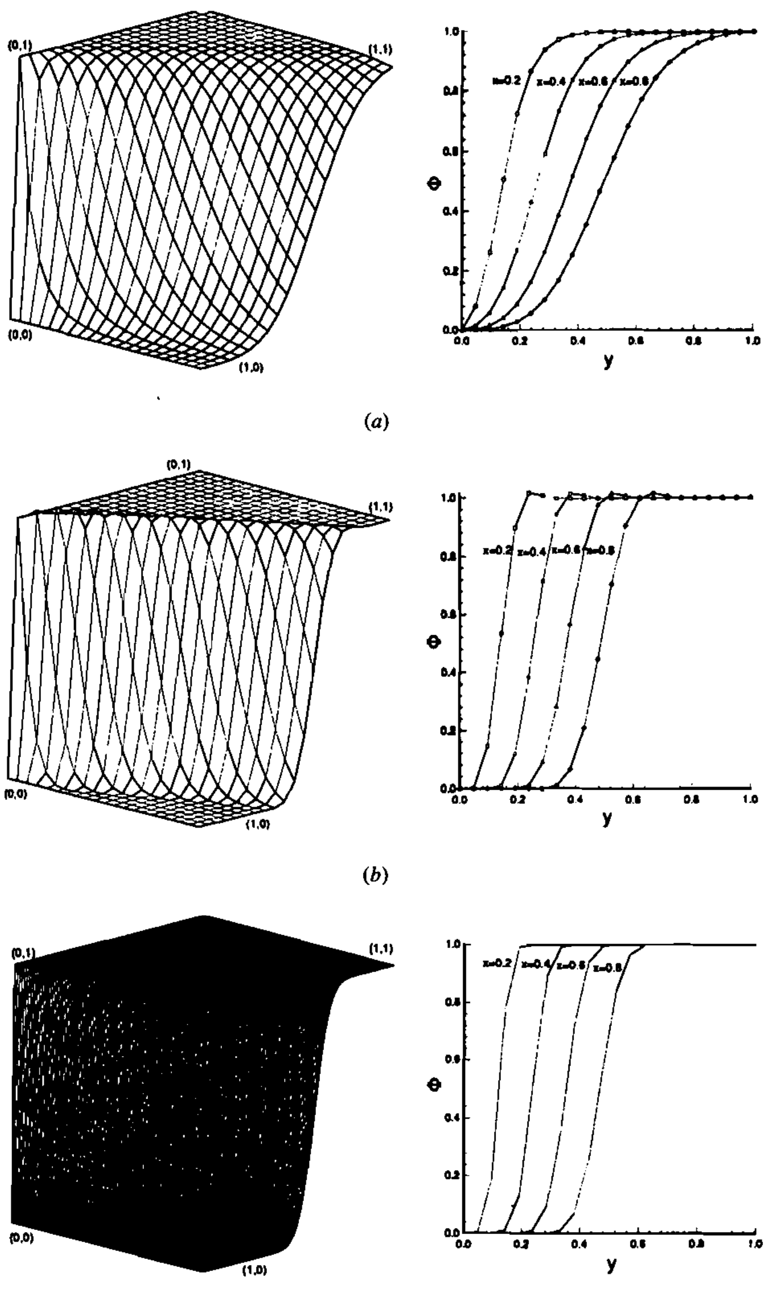


Figure 14. Finite-element solutions of Φ and their distributions at $x = \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$ in the case of $\mu = 2 \times 10^{-3}$: (*a*) characteristic version implemented at 20 × 20; (*b*) characteristic-free version implemented at 20 × 20; (*c*) characteristic-free version implemented at 150 × 150.

smooth behavior clearly demonstrates that nonphysical spatial oscillations are not exhibited under certain conditions. With increasing Peclet numbers or decreasing diffusivities, poor performance in the case of $\mu = 2 \times 10^{-3}$ makes such a method hardly applicable to problems involving steep gradients of the convected field variable Φ. To suppress these wiggles, one can of course keep reducing the mesh size until the corresponding Peclet number falls into the category of the monotonic region in Figure 12. There is considerable computational expense associated with the frontal solver. This is a consequence of having to adopt other alternatives. Here, we adopt the characteristic functionality to make the discrete system monotonic. In the presence of characteristic enhancement, the monotonic solution for the case plotted in Figure 12 can also be obtained in a domain having much fewer nodal points. Figure 13 clearly illustrates the importance of incorporating the characteristic information into the proposed upwind finite-element method. With the diffusivity set equal to 10^{-6}, the diffusion effects are virtually eliminated. A solution underlying the characteristic upwind model is smoothly exhibited in Figure 14. Furthermore, by setting the diffusivity coefficient equal to 10^{-6}, the cell Reynolds number approaches infinity. The case considered is known as the conventional skew advection problem. While these computed solutions in situations where convection significantly dominates diffusion are smoothly predicted, the quality of the analysis suffers from excessive numerical diffusion. In summary, the smeared solution shown in Figure 14 is more or less representative of the outcome of any method that is restricted by the M-matrix constraint condition.

CONCLUSION

In this article we have calculated finite-element solutions for the convection-diffusion equation defined in a two-dimensional domain. To stabilize this differential system in regions of high gradients, an oscillation suppressant is incorporated into the Petrov-Galerkin framework. Thanks to the underlying **M** matrix, which has been implemented as a crucial stabilizing ingredient in Petrov-Galerkin weighted-residuals methods, we can judiciously determine the amount and type of artificial diffusion a priori which is necessary to resolve overshoots or undershoots. We have endeavored to conduct basic studies with the focus on numerical accuracy and stability. By virtue of the modified equation analysis, considerable insight into the behavior of the consistency property has been obtained. An attribute of the proposed streamline method is its stabilization along the flow direction. It is worthwhile to note that the employed monotonicity property is applicable to multiple dimensions. This finite-element model has been verified by solving problems with analytic solutions of different characters. The success of the proposed upwind finite-element model provides a strong impetus for utilization of the method in the simulation of problems in fluid mechanics.

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