科目名稱：有限元素流體計算方法

課　號：525M2070　　　授課教師　許文翰（TEL.: 702-5170）

每週時數：　3　　學分　3　（全年或半年）　半年　（必選）　選修

Mon. 9:10 ~ 12:00 am

課程大綱：

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1. Fundamentals of finite element methods.
2. Galerkin finite element method for incompressible Navier-stokes flows.
3. Upwinding finite element methods for convection dominated viscous flows.
4. Tayler Galerkin finite element method for fluid problems
5. characteristic finite element methods for hyperbolic system.
6. Flux corrected transport finite element method (FCT-FEM) for hyperbolic conservation law system.

教　材：

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1. Finite Element Programming of the Navier-Stokes Equations, by C.Taylor and T.G. Hughes.
2. Technical papers

Office Hour: Wensday 1:10 pm ~ 5:00 pm

Credits: Four projects (60%)
Final report (30%)
Homework, Quiz, ~~partipation~~ (10%)

525 M 2070

有限元素流体計算方法

Monday 1, 2, 3. (許文翰)

- Introduction
- Method of weighted Residual Method*
- Collocation Finite Element Method.
- Upwind Finite Element Method.
- Modified Weighted Residual Method
- Variable Upwinding and Adaptive Mesh Refinement Method

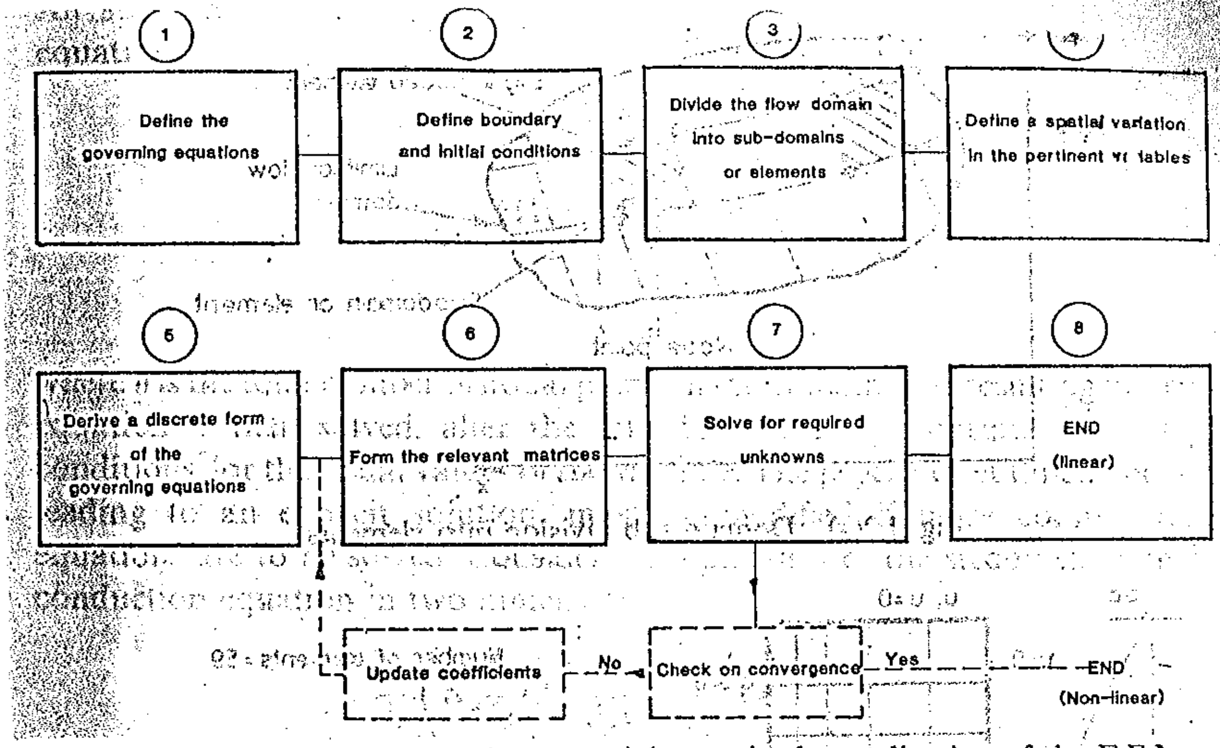
} convective transport problems

- Taylor-Galerkin Method for Convective Transport Problems
- Petrov-Galerkin Methods for Time Dependent Convective Transport Equation

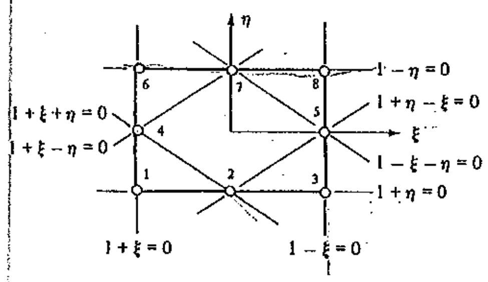
} Time-Dependent Transport Problems

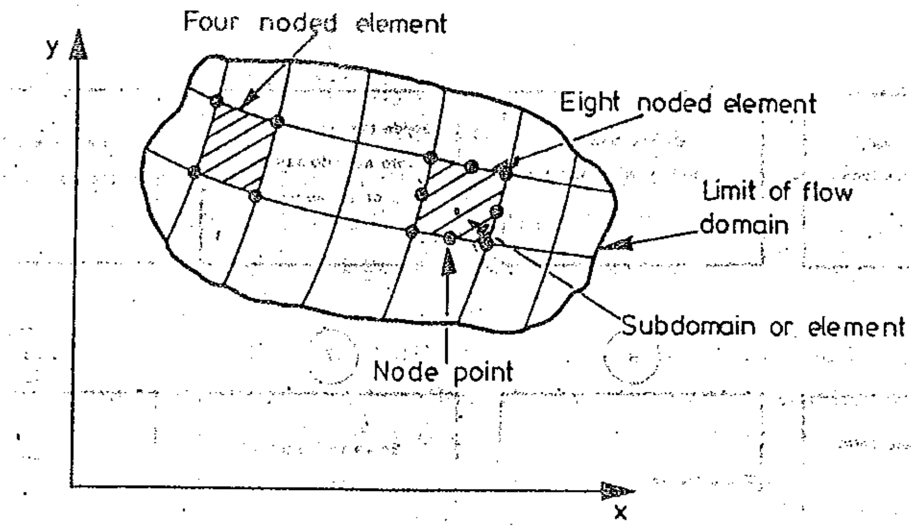
- ω-ψ Finite Element methods for Stokes and Incompressible ${}_{N-S}$ Flows
- Lagrangian Finite Element method for N-S Flows
- Frontal Equation solver.

* Finite Element Programming of the Navier-Stokes Equations by C. Taylor, T. G. Hughes

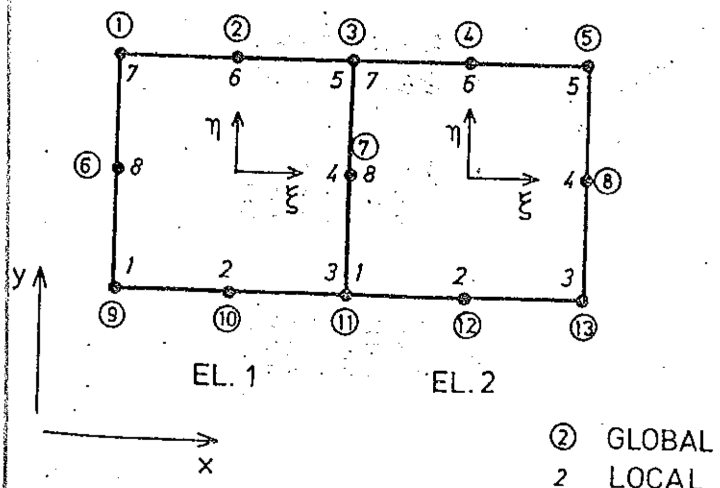


Block diagram showing essential steps in the application of the F.E.M.

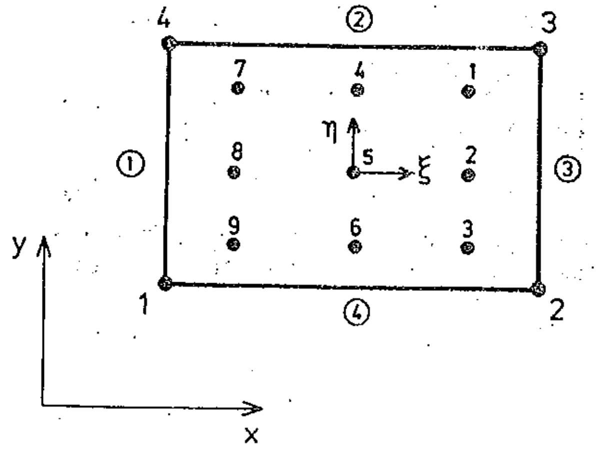




Domain sub division into elements



Local and global node numbering for element nodes



Finite element method:

A numerical technique using variational methods and interpolation theory for solving engineering problems in differential or integral forms.
ie, The finite element method is a piecewise application of a variational method

- General, systematic, modular
- Complicated geometries.
- modeling and simulating most physical phenomena

Basic features:

① Discretization of the domain:
Any complicated domain is divided into a number of simpler subdomains (finite elements).

② Element-wise polynomial approximation:
Over each element, the solution of the equation being solved is approximated by interpolation polynomials.

Note

- There is only one finite-element method which is characterized by the above two features.
- The techniques used to determine the nodal values can be one of the several variational methods (including the method of Ritz, Galerkin, least squares, collocation etc).

→ Finite element model:

It refers to the final algebraic equations obtained after applying the finite-element method and a variational method to the equation being solved.

Basic steps in the finite-element analysis

1. Discretization of a domain
   ⓐ generating finite element mesh (製作網格)
   ⓑ generating connectivity 連結 matrix (A relation indicating the local and global element location)

2. Approximation of the solution (over each element)
   ⓐ Derivation of the shape functions (interpolation functions) over each element such that the solution variables and independent spatial variables can be expressed in terms of its nodal values
   ⓑ Employing a variational method to derive the algebraic equations for the unknowns in terms of nodal values.

3. Assembly of elements.
   Combine the algebraic equations of all elements in the mesh by imposing the continuity of the primary nodal variables → obtaining a algebraic equations govering the whole problems.

4. Imposition of boundary conditions. (Dirichlet type B.C.)
5. Solution of equations
6. Post-processor.
   Using the nodal values of the primary variables to compute the secondary variables.

Nomenclature:

finite element mesh —— Number the nodes and elements of the collection.

Connectivity —— the procedure of putting the elements together.

connectivity matrix —— the relationship of a elemet nodes between the local and global positions.

Trial function or Test function $\hat{u}$ (shape function, interpolating function, basis function) $\phi_i(\cdot)$
Weighting function. $u(\cdot) = \hat{u}(\cdot) \approx \sum u_i \phi_i(\cdot)$

Note: The basis functions are formally required to be members of a complete set of functions.

Note: The choices of basis functions

- Linear polynomial bases
- Quadratic polynomial bases
- Cubic polynomial bases
- Lagrange polynomial bases $l_i^n(\xi) = \dfrac{(\xi-\xi_0)\cdots(\xi-\xi_{i-1})(\xi-\xi_{i+1})\cdots(\xi-\xi_n)}{(\xi_i-\xi_0)\cdots(\xi_i-\xi_{i-1})(\xi_i-\xi_{i+1})\cdots(\xi_i-\xi_n)}$
- Hermite polynomial bases
- Serendipity bases

The choices of weighting functions

- Galerkin method
- Subdomain method
- Collocation method

Two dimensional basis functions

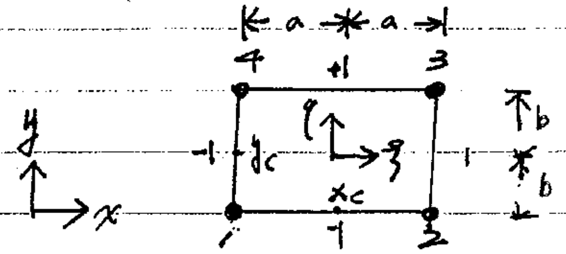
- Lagrangian basis functions — the product of two one-dimensional bases (could be linear, quadratic, cubic, etc)
- Serendipity ~~Basic~~ basic functions — preserve important properties of Lagrangian basis but decrease the d.o.f by eliminating interior nodes.

  Serendipity polynomial

Shape functions

(1) For 4 nodal element

$x = x(\xi, \eta)$
$y = y(\xi, \eta)$



$$\xi = (x - x_c)/a \quad , \quad d\xi = dx/a$$
$$\eta = (y - y_c)/b \quad , \quad d\eta = dy/b$$

bilinear

Assume $\quad \phi = \alpha_1 + \alpha_2 \xi + \alpha_3 \eta + \alpha_4 \xi \eta$

$$= \begin{pmatrix} N_1 & N_2 & N_3 & N_4 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = \sum_{i=1}^{4} N_i \phi_i$$

$$\rightarrow \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = \begin{pmatrix} 1 & \xi_1 & \eta_1 & \xi_1\eta_1 \\ 1 & \xi_2 & \eta_2 & \xi_2\eta_2 \\ 1 & \xi_3 & \eta_3 & \xi_3\eta_3 \\ 1 & \xi_4 & \eta_4 & \xi_4\eta_4 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix}$$

or $\quad \underline{\phi} = \underline{C}\, \underline{\alpha}$

or $\quad \underline{\alpha} = \underline{C}^{-1} \underline{\phi}$

where $\quad \underline{C}^{-1} = \dfrac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$

ie, $\quad \alpha_1 = \frac{1}{4}(\phi_1 + \phi_2 + \phi_3 + \phi_4) \quad ; \quad \alpha_2 = \frac{1}{4}(-\phi_1 + \phi_2 + \phi_3 - \phi_4)$

$\quad \alpha_3 = \frac{1}{4}(-\phi_1 - \phi_2 + \phi_3 + \phi_4) \quad ; \quad \alpha_4 = \frac{1}{4}(\phi_1 - \phi_2 + \phi_3 - \phi_4)$

$$\rightarrow \quad N_1 = \frac{1}{4}(1-\xi-\eta+\xi\eta) = \frac{1}{4}(1-\xi)(1-\eta)$$

$$N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

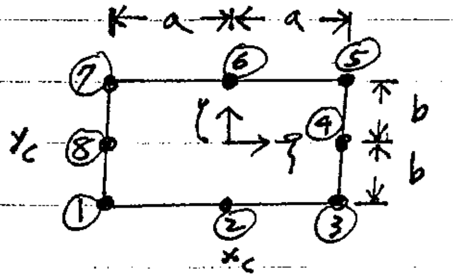
$$N_3 = \frac{1}{4}(1+\xi)(1+\eta)$$

$$N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

or

$$N_i = \frac{1}{4}(1+\xi_i\xi)(1+\eta_i\eta) \qquad i=1,2,3,4$$

(2) For 8-nodal element



Corner nodes 1, 3, 5, 7

$$N_i = \frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i)(\xi\xi_i+\eta\eta_i-1)$$

Midside nodes 2, 4, 6, 8

For 2, 6 nodal points

$$N_i = \frac{1}{2}(1-\xi^2)(1+\eta\eta_i)$$

For 4, 8 nodal points

$$N_i = \frac{1}{2}(1+\xi\xi_i)(1-\eta^2)$$

Where the polynomial used in this case is of the form,

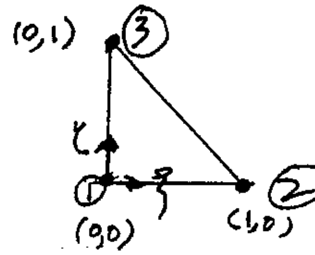
$$\phi = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy + \alpha_5 x^2 + \alpha_6 y^2 + \alpha_7 xy^2 + \alpha_8 x^2 y$$

Typical finite element

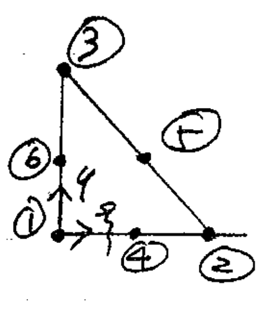
(一) master triangle

(i) For linear shape function

$$N_1(\xi,\eta) = 1-\xi-\eta$$
$$N_2(\xi,\eta) = \xi$$
$$N_3(\xi,\eta) = \eta$$

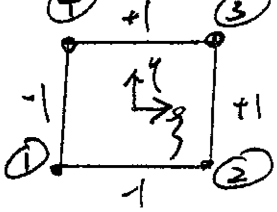


(ii) For quadratic shape function

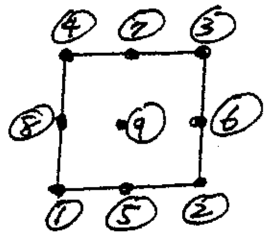


$$N_1(\xi,\eta) = 2(1-\xi-\eta)\left(\tfrac{1}{2}-\xi-\eta\right)$$
$$N_2(\xi,\eta) = 2\xi(\xi-\tfrac{1}{2})$$
$$N_3(\xi,\eta) = 2\eta(\eta-\tfrac{1}{2})$$
$$N_4(\xi,\eta) = 4\xi(1-\xi-\eta)$$
$$N_5(\xi,\eta) = 4\xi\eta$$
$$N_6(\xi,\eta) = 4\eta(1-\xi-\eta)$$

(二) (i) For Bilinear square



$$N_1(\xi,\eta) = \tfrac{1}{4}(1-\xi)(1-\eta)$$
$$N_2(\xi,\eta) = \tfrac{1}{4}(1+\xi)(1-\eta)$$
$$N_3(\xi,\eta) = \tfrac{1}{4}(1+\xi)(1+\eta)$$
$$N_4(\xi,\eta) = \tfrac{1}{4}(1-\xi)(1+\eta)$$



(ii) For Biquadratic square

$$N_1(\xi,\eta) = \tfrac{1}{4}\xi\eta(1-\xi)(1-\eta)$$
$$N_2(\xi,\eta) = -\tfrac{1}{4}\xi\eta(1-\xi)(1-\eta)$$
$$N_3(\xi,\eta) = \tfrac{1}{4}\xi\eta(1+\xi)(1+\eta)$$
$$N_4(\xi,\eta) = -\tfrac{1}{4}\xi\eta(1-\xi)(1+\eta)$$
$$N_5(\xi,\eta) = -\tfrac{1}{2}\eta(1-\xi^2)(1-\eta)$$
$$N_6(\xi,\eta) = \tfrac{1}{2}\xi(1+\xi)(1-\eta^2)$$
$$N_7(\xi,\eta) = \tfrac{1}{2}\eta(1+\eta)(1-\xi^2)$$
$$N_8(\xi,\eta) = -\tfrac{1}{2}\xi(1-\xi)(1-\eta^2)$$
$$N_9(\xi,\eta) = (1-\xi^2)(1-\eta^2)$$

Hermite polynomials are cubic splines that display, in addition to second-order derivative continuity (C^2) over the element, first-order derivative continuity (C^1) between elements.

Hermite basis functions :

The test functions v assume the form

$$v(x) = \sum_{i=1}^{N} v_i \, \phi_i^0(x) + \sum_{i=1}^{N} v_i' \, \phi_i^1(x)$$

Where N is the no. of nodes in the mesh and basis functions ϕ_i^0, ϕ_i^1 have the properties

- $\phi_i^0(x_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

  $\dfrac{d\phi_i^0(x_j)}{dx} = 0$ $\qquad 1 \leq i, j \leq N \qquad (1)$

- $\phi_i^1(x_j) = 0$

  $\dfrac{d\phi_i^1(x_j)}{dx} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

$\Rightarrow$ $v(x_i) = \sum_{i=1}^{N} v_i \, \phi_i^0(x_i) + \sum_{i=1}^{N} v_i' \, \phi_i^1(x_i) = v_i$

$\dfrac{dv(x_i)}{dx} = \sum_{i=1}^{N} v_i \, \phi_{i,x}^0(x_i) + \sum_{i=1}^{N} v_i' \, \phi_{ix}'(x_i) = v_i'$

for $i = 1, 2, \dots, N$

Basis functions, such as those in equation (1), which interpolate derivatives as well as values at nodes, are called Hermite basis functions.

First order Hermite basis funtions are Hermite cubic basis functions.

$$v(x) = c_1 + c_2 x + c_3 x^2 + c_4 x^3$$

$$\phi_1^0 = \phi(x_e) = c_1 + c_2 x_e + c_3 x_e^2 + c_4 (x_e)^3$$

$$\phi_2^0 = \phi(x_{e+1}) = c_1 + c_2 x_{e+1} + c_3 x_{e+1}^2 + c_4 x_{e+1}^3$$

$$\phi_1^1 = + \frac{\partial \phi}{\partial x}(x_e) = + c_2 + 2 c_3 x_e + 3 c_4 x_e^2$$

$$\phi_2^1 = + \frac{\partial \phi}{\partial x}(x_{e+1}) = + c_2 + 2 c_3 x_{e+1} + 3 c_4 x_{e+1}^2$$

$$\rightarrow v(x) = c_1 + c_2 x + c_3 x^2 + c_4 x^4$$

$$= v_1 \phi_1^0 + v_2 \phi_2^0 + v_1' \phi_1^1 + v_2' \phi_2^1$$

where

$$\phi_1^0 = 1 - 3\left(\frac{x - x_e}{h_e}\right)^2 + 2\left(\frac{x - x_e}{h_e}\right)^3$$

$$\phi_2^0 = 3\left(\frac{x - x_e}{h_e}\right)^2 - 2\left(\frac{x - x_e}{h_e}\right)^3$$

$$\phi_1^1 = +(x - x_e)\left(1 - \frac{x - x_e}{h_e}\right)^2$$

$$\phi_2^1 = +(x - x_e)\left[\left(\frac{x - x_e}{h_e}\right)^2 - \frac{x - x_e}{h_e}\right]$$

$v_1 = v(x_e)$

$v_2 = v(x_{e+1})$

$v_1' = v_{,x}(x_e)$

$v_2' = v_{,x}(x_{e+1})$

Change of coordinate from x to ξ by

$$\xi = \frac{2(x - x_e)}{h_e} - 1$$

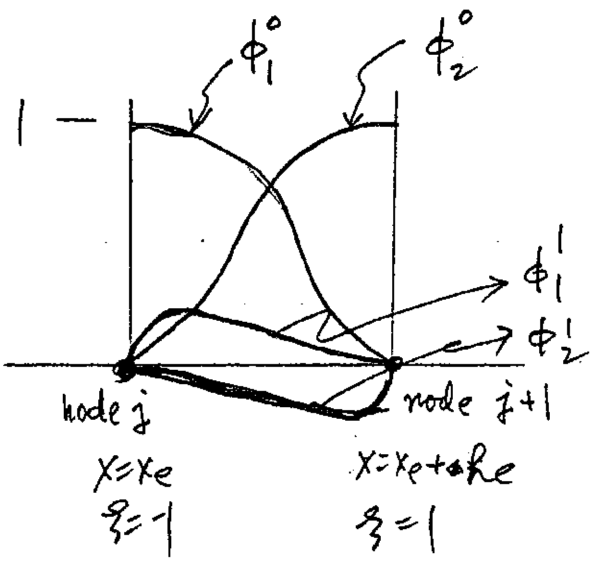
to transform $[x_e, x_{e+1}]$ to $[-1, 1]$

$$\Rightarrow \phi_1^0(\xi) = \frac{1}{4}(\xi-1)^2(\xi+2)$$

$$\phi_2^0(\xi) = -\frac{1}{4}(\xi+1)^2(\xi-2)$$

$$\phi_1^1(\xi) = \frac{1}{8}h_e(\xi+1)(\xi-1)^2$$

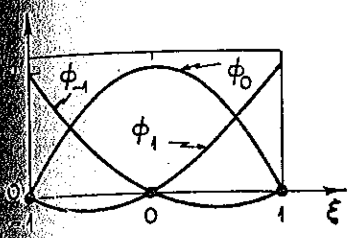
$$\phi_2^1(\xi) = \frac{1}{8}h_e(\xi-1)(\xi+1)^2$$



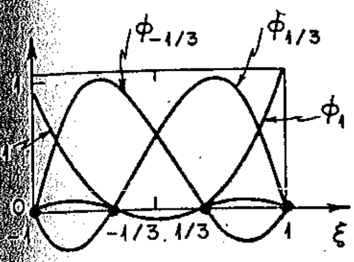
$\Rightarrow$ Hermite cubic basis provides a continuously differentiable interpolation scheme for functions (v) whose values and first derivatives are known at the nodes (v_i, v_i')

Shape function

Polynomial basis function



(a)



(b)

Quadratic (*a*) and cubic (*b*). basis functions defined in local ξ coordinate system.

Linear, Quadratic, and Cubic Basis Functions Defined in the Dimensionless ξ Coordinate System

Degree	Function	Form $(-1 \leq \xi \leq 1)$
Linear	$\phi_{-1}(\xi)$	$\frac{1}{2}(1-\xi)$
	$\phi_1(\xi)$	$\frac{1}{2}(1+\xi)$
Quadratic	$\phi_{-1}(\xi)$	$-\frac{1}{2}\xi(1-\xi)$
	$\phi_0(\xi)$	$1-\xi^2$
	$\phi_1(\xi)$	$\frac{1}{2}\xi(1+\xi)$
Cubic	$\phi_{-1}(\xi)$	$\frac{1}{16}(-9\xi^3+9\xi^2+\xi-1)$ or $\frac{1}{16}(1-\xi)(9\xi^2-1)$
	$\phi_{-1/3}(\xi)$	$\frac{9}{16}(3\xi^3-\xi^2-3\xi+1)$ or $\frac{9}{16}(3\xi-1)(\xi^2-1)$
	$\phi_{1/3}(\xi)$	$\frac{9}{16}(-3\xi^3-\xi^2+3\xi+1)$ or $-\frac{9}{16}(3\xi+1)(\xi^2-1)$
	$\phi_1(\xi)$	$\frac{1}{16}(9\xi^3+9\xi^2-\xi-1)$ or $\frac{1}{16}(1+\xi)(9\xi^2-1)$

Lagrangian, Serendipity, Hermite basis function

Basis Functions Formulated Using Quadratic, Cubic, and Hermitian Cubic Polynomials[a]

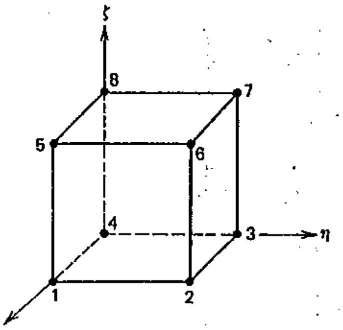
	Lagrangian	Serendipity
Linear	$\frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i)$	$\frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i)$
Quadratic		
Corner node	$\frac{1}{4}\xi\xi_i(1+\xi\xi_i)\eta\eta_i(1+\eta\eta_i)$	$\frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i)(\xi\xi_i+\eta\eta_i-1)$
Side node, $\xi_i=0$	$\frac{1}{2}\eta\eta_i(1+\eta\eta_i)(1-\xi^2)$	$\frac{1}{2}(1-\xi^2)(1+\eta\eta_i)$
Side node, $\eta_i=0$	$\frac{1}{2}\xi\xi_i(1+\xi\xi_i)(1-\eta^2)$	$\frac{1}{2}(1+\xi\xi_i)(1-\eta^2)$
Interior node	$(1-\xi^2)(1-\eta^2)$	—
Cubic		
Corner node	$\frac{1}{256}(9\xi^2-1)(\xi_i\xi+1)(9\eta^2-1)(\eta_i\eta+1)$	$\frac{1}{32}(1+\xi\xi_i)(1+\eta\eta_i)[9(\xi^2+\eta^2)-10]$
Side node $\begin{cases}\eta=\pm1, \xi=\pm\frac{1}{3}\\ \eta=\pm\frac{1}{3}, \xi=\pm1\end{cases}$	$\frac{9}{256}(9\eta^2-1)(\eta_i\eta+1)(1-\xi^2)(1+9\xi\xi_i)$ $\frac{9}{256}(9\xi^2-1)(\xi_i\xi+1)(1-\eta^2)(1+9\eta\eta_i)$	$\frac{9}{32}(1-\xi^2)(1+9\xi\xi_i)(1+\eta\eta_i)$
Interior node, $\xi=\pm\frac{1}{3}, \eta=\pm\frac{1}{3}$	$\frac{81}{256}(1-\xi^2)(1-\eta^2)(1+9\xi\xi_i)(1+9\eta\eta_i)$	—
Hermite		
ϕ^1_{00i}	$\frac{1}{16}(\xi+\xi_i)^2(\xi\xi_i-2)(\eta+\eta_i)^2(\eta\eta_i-2)$	$\frac{1}{8}(1+\xi\xi_i)(1+\eta\eta_i)(2+\xi\xi_i+\eta\eta_i-\xi^2-\eta^2)$
ϕ^1_{10i}	$-\frac{1}{16}\xi_i(\xi+\xi_i)^2(\xi\xi_i-1)(\eta+\eta_i)^2(\eta\eta_i-2)$	$-\frac{\xi_i}{8}(1-\xi^2)(1+\xi\xi_i)(1+\eta\eta_i)$
ϕ^1_{01i}	$-\frac{1}{16}(\xi+\xi_i)^2(\xi\xi_i-2)\eta_i(\eta+\eta_i)^2(\eta\eta_i-1)$	$-\frac{\eta_i}{8}(1-\eta^2)(1+\xi\xi_i)(1+\eta\eta_i)$
ϕ^1_{11i}	$\frac{1}{16}\xi_i(\xi+\xi_i)^2(\xi\xi_i-1)\eta_i(\eta+\eta_i)^2(\eta\eta_i-1)$	—

Serendipity element – rectangular element which has no interior nodes.
- interpolation functions can not be obtained using tensor products of 1-D interpolation functions.

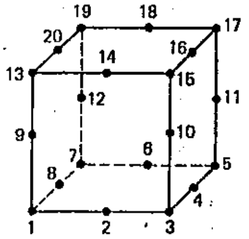
Lagrangian element – interpolation functions for these elements can be obtained from 1-D interpolation functions

Basis Functions for Serendipity-Type C^0 Hexahedral Elements; ξ_i, η_i, and ζ_i Are the Local Coordinates of the ith Point

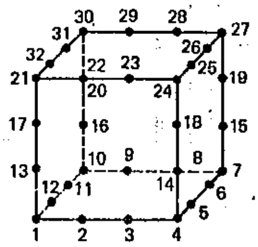
C^0 Elements



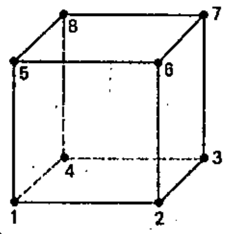
$$\phi_i=\frac{1}{8}(1+\xi\xi_i)(1+\eta\eta_i)(1+\zeta\zeta_i)\ (i=1,2,\ldots,8)$$



$$\phi_i=\frac{1}{8}(1+\xi\xi_i)(1+\eta\eta_i)(1+\zeta\zeta_i)(\xi\xi_i+\eta\eta_i+\zeta\zeta_i-2)\ (i=1,3,5,7,13,15,17,19)$$
$$\phi_i=\frac{1}{4}(1-\zeta^2)(1+\xi\xi_i)(1+\eta\eta_i)\ (i=9,10,11,12)$$
$$\phi_i=\frac{1}{4}(1-\xi^2)(1+\eta\eta_i)(1+\zeta\zeta_i)\ (i=8,4,16,20)$$
$$\phi_i=\frac{1}{4}(1-\eta^2)(1+\xi\xi_i)(1+\zeta\zeta_i)\ (i=2,6,18,14)$$



$$\phi_i=\frac{1}{64}(1+\xi\xi_i)(1+\eta\eta_i)(1+\zeta\zeta_i)[9(\xi^2+\eta^2+\zeta^2)-19]\ (i=1,4,7,10,21,24,27,30)$$
$$\phi_i=\frac{9}{64}(1-\xi^2)(1+9\xi\xi_i)(1+\eta\eta_i)(1+\zeta\zeta_i)\ (i=11,12,5,6,31,32,26,25)$$
$$\phi_i=\frac{9}{64}(1-\eta^2)(1+9\eta\eta_i)(1+\xi\xi_i)(1+\zeta\zeta_i)\ (i=2,3,8,9,22,23,28,29)$$
$$\phi_i=\frac{9}{64}(1-\zeta^2)(1+9\zeta\zeta_i)(1+\xi\xi_i)(1+\eta\eta_i)\ (i=13,8,15,16,17,18,19,20)$$



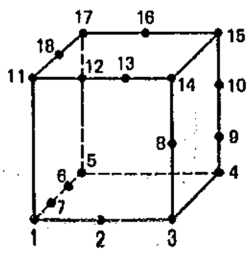
C^0 Hermite[a]

$$\phi_{000i}=\frac{1}{16}(1+\xi\xi_0)(1+\eta\eta_0)(1+\zeta\zeta_0)(2+\xi\xi_0+\eta\eta_0+\zeta\zeta_0-\xi^2-\eta^2-\zeta^2)$$
$$\phi_{100i}=-\frac{\xi_0}{16}(1-\xi^2)(1+\xi\xi_0)(1+\eta\eta_0)(1+\zeta\zeta_0)$$
$$\phi_{010i}=-\frac{\eta_0}{16}(1-\eta^2)(1+\xi\xi_0)(1+\eta\eta_0)(1+\zeta\zeta_0)$$
$$\phi_{001i}=-\frac{\zeta_0}{16}(1-\zeta^2)(1+\xi\xi_0)(1+\eta\eta_0)(1+\zeta\zeta_0)$$

Mixed C^0 element

Corner nodes

$$\phi_i=\alpha_i\beta_i$$
$$\alpha_i=\tfrac{1}{8}(1+\xi\xi_i)(1+\eta\eta_i)(1+\zeta\zeta_i)$$
$$\beta_i=\beta_\xi+\beta_\eta+\beta_\zeta$$



where

Side	β_ξ	β_η	β_ζ
Linear	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
Quadratic	$\xi\xi_i-\frac{2}{3}$	$\eta\eta_i-\frac{2}{3}$	$\zeta\zeta_i-\frac{2}{3}$
Cubic	$\frac{9}{8}\xi^2-\frac{19}{24}$	$\frac{9}{8}\eta^2-\frac{19}{24}$	$\frac{9}{8}\zeta^2-\frac{19}{24}$

Edge nodes

Quadratic

$$\phi_i=\tfrac{1}{4}(1-\xi^2)(1+\eta\eta_i)(1+\zeta\zeta_i)\ (i=12)$$
$$\phi_i=\tfrac{1}{4}(1+\xi\xi_i)(1-\eta^2)(1+\zeta\zeta_i)\ (i=18)$$
$$\phi_i=\tfrac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i)(1-\zeta^2)\ (i=2)$$

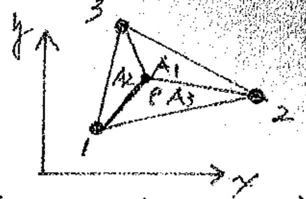
Cubic

$$\phi_i=\tfrac{9}{64}(1-\xi^2)(1+9\xi\xi_i)(1+\eta\eta_i)(1+\zeta\zeta_i)\ (i=6,7)$$
$$\phi_i=\tfrac{9}{64}(1+\xi\xi_i)(1-\eta^2)(1+9\eta\eta_i)(1+\zeta\zeta_i)\ (i=12,13)$$
$$\phi_i=\tfrac{9}{64}(1+\xi\xi_i)(1+\eta\eta_i)(1-\zeta^2)(1+9\zeta\zeta_i)\ (i=9,10)$$

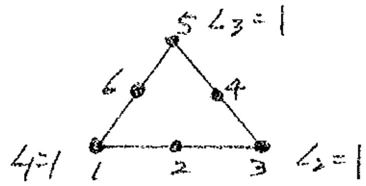
[a] M. Th. van Genuchten, unpublished manuscript.
From Verge 1975

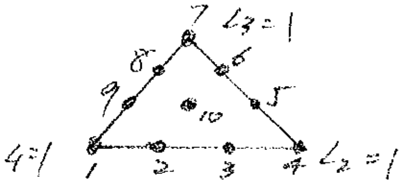
Triangular element

- 2-dimensional



$L_i = A_i/A \quad (i=1,2,3)$





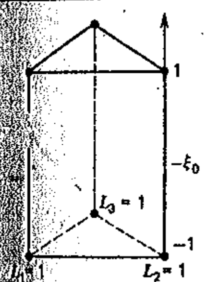
Linear, Quadratic, and Cubic Basis Functions for Triangular Elements

Basis	Functions: Linear	Quadratic	Cubic
ϕ_1	L_1	$2L_1^2 - L_1$	$\frac{1}{2}L_1(3L_1-1)(3L_1-2)$
ϕ_2	L_2	$4L_1L_2$	$\frac{9}{2}L_1L_2(3L_1-1)$
ϕ_3	L_3	$2L_2^2 - L_2$	$\frac{9}{2}L_1L_2(3L_2-1)$
ϕ_4	—	$4L_2L_3$	$\frac{1}{2}L_2(3L_2-1)(3L_2-2)$
ϕ_5	—	$2L_3^2 - L_3$	$\frac{9}{2}L_2L_3(3L_2-1)$
ϕ_6	—	$4L_3L_1$	$\frac{9}{2}L_2L_3(3L_3-1)$
ϕ_7	—	—	$\frac{1}{2}L_3(3L_3-1)(3L_3-2)$
ϕ_8	—	—	$\frac{9}{2}L_3L_1(3L_3-1)$
ϕ_9	—	—	$\frac{9}{2}L_3L_1(3L_1-1)$
ϕ_{10}	—	—	$27L_1L_2L_3$

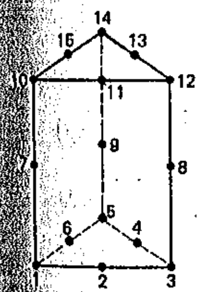
- Three-dimensional

Basis Functions for Serendipity-Type Pentahedral Elements and Tetrahedral Elements

Pentahedral elements



$$\phi_i = \tfrac{1}{2}(1+\xi\xi_i)(L_i)$$



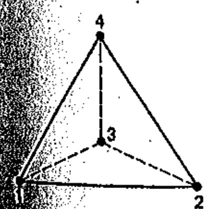
$$\phi_i = \tfrac{1}{2}L_i(2L_i-1)(1+\xi\xi_i) - \tfrac{1}{2}L_i(1-\xi^2) \quad (i=1,3,5,10,12,14)$$

$$\phi_i = 2L_jL_k(1+\xi\xi_i) \quad (i=2,4,6,11,13,15)$$

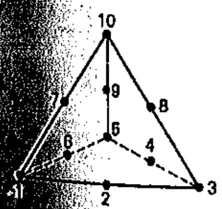
$$\phi_i = L_i(1-\xi^2) \quad (i=7,8,9)$$

where j and k are nodes located along the same edge as i.

Tetrahedral elements



$$\phi_i = L_i$$



$$\phi_i = (2L_i-1)L_i \quad (i=1,3,10)$$

$$\phi_i = 4L_jL_k \quad (i=2,4,6,7,8,9)$$

where j and k are nodes located along the same edge as i.

From Zienkiewicz, 1971.

Isoparametric elements :

Isoparametric elements are those which both geometry of the element and the dependent variables can be interpolated by the same functions in terms of natural coordinates (ξ, η).

i.e $x = \sum_{i=1}^{N} N_i(\xi, \eta)\, x_i$

$y = \sum_{i=1}^{N} N_i(\xi, \eta)\, y_i$

$u = \sum_{i=1}^{N} N_i(\xi, \eta)\, u_i$

where N: no. of nodal points within the elements

Numerical integration

Gaussian quadrature integration formula

$$I = \int_{-1}^{1} f(\xi)\, d\xi = \sum_{i=1}^{m} a_i f(\xi_i)$$

where

m: total no. of integration points

a_i : i-th weighting factor

ξ_i : coordinate of i-th integration point

Weighting factors and Gaussian sampling point positions

m	i	a_i	ξ_i
1	1	2	0
2	1	1	$-1/\sqrt{3}$
	2	1	$1/\sqrt{3}$

m	i	a_i	ξ_i
3	1	$5/9$	$-(0.6)^{1/2}$
	2	$8/9$	0
	3	$5/9$	$(0.6)^{1/2}$
4	1	$0.5 - \sqrt{30}/36$	$-\left(\frac{3+\sqrt{4.8}}{7}\right)^{1/2}$
	2	$0.5 - \sqrt{30}/36$	$\left(\frac{3+\sqrt{4.8}}{7}\right)^{1/2}$
	3	$0.5 + \sqrt{30}/36$	$-\left(\frac{3-\sqrt{4.8}}{7}\right)^{1/2}$
	4	$0.5 + \sqrt{30}/36$	$\left(\frac{3-\sqrt{4.8}}{7}\right)^{1/2}$

$$I = \int_{-1}^{1}\int_{-1}^{1} F(\xi, \eta)\, d\xi\, d\eta$$

$$= \sum_{j=1}^{m'} a_j \sum_{i=1}^{m} a_i F(\xi_i, \eta_j)$$

$$= \sum_{j=1}^{m'} \sum_{i=1}^{m} a_j a_i F(\xi_i, \eta_j)$$

A polynomial of degree n is integrated exactly by employing $(n+1)/2$ Gauss integration points

Consider $\partial\phi/\partial t + \underline{u}\cdot\underline{\nabla}\phi = \nabla\cdot(k\,\underline{\nabla}\phi)$ in Ω

parabolic p.d.e. $\partial\phi/\partial n + a\,\phi = \gamma$ on $\partial\Omega$

$\phi(\underline{x}, 0) = f(\underline{x})$ in Ω

- Semidiscrete finite element approximation:

Weighted-residual statement

\+ semidiscrete finite element method

$$\phi_h(\underline{x}, t) = \sum_{i=1}^{N} \phi_j(t) N_j(\underline{x})$$

$\longrightarrow$ A system of ordinary differential equations

$$\begin{cases} \underline{B}\,\dfrac{d\underline{\phi}}{dt} + \underline{A}\,\underline{\phi} = \underline{C} \\ \text{with initial data} \\ \underline{B}\,\underline{\phi}(0) = \underline{\phi}_0 \end{cases}$$

The solution of above semidiscrete system involves the consideration of time discretization scheme, accuracy, and stability.

Time discretization:

(i) The simplest and most widely used methods for integrating above system are constructed by differencing with respect to time.

(ii) The integration of semidiscrete system such as collocation on finite element.

Stability:

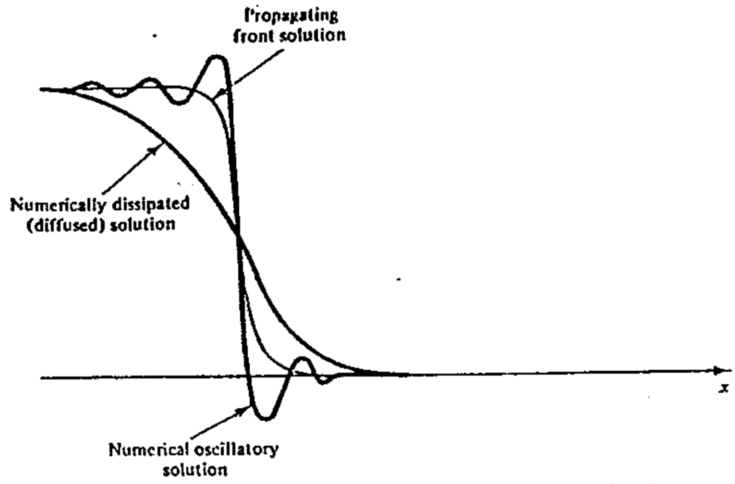
The stability properties differ depending on the choice of elements, method, and integration (time) formula. It is made by performing Von Neumann stability analysis.

<u>accuracy</u>:

(i) dissipation error:
(Amplitude error)

(ii) Phase error:
(Dispersion error)

- propagation of short wavelengths out of phase tends to promote oscillation.
- Harmonics of small wave lengths are important in resoluting sharp fronts.

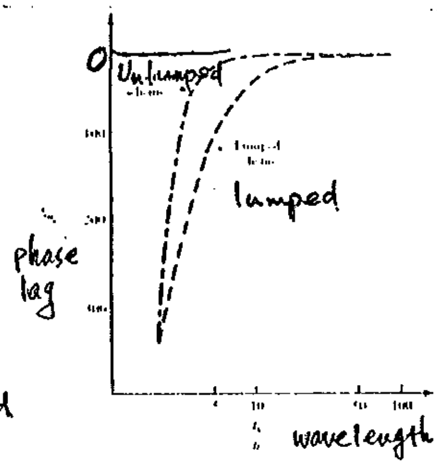


Oscillatory and dissipative numerical approximation near a propagating front.

Note: • The inability of a given numerical scheme to propagate individual frequencies without phase errors and with accurate amplitudes leads to oscillations.

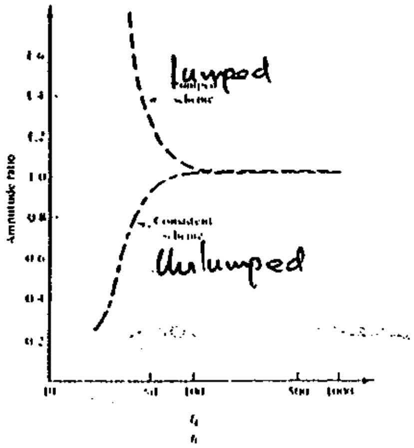
• A qualitative evaluation of schemes for a given problem is possible through a consideration of the spectrum of eigenvalues on $\underline{B}^{-1}\underline{A}$.

- The eigenvalues λ_i of $(\underline{B}^{-1}\underline{A})$ may be complex if convection dominates diffusion. The imaginary parts of these eigenvalues causes the presence of oscillatory ripples.
- small wavelengths are propagated with larger phase lags than the longer wavelength components.
- Unlumped methods are superior to the corresponding lumped methods insofar as phase accuracy is concerned



Graph of phase lag δ against wavelength per unit length l_s/h for consistent and lumped Crank–Nicolson type schemes (From Gray and Pinder [1976].)

- the amplitude ratio at short wavelength region is shown in this plot.
- The removal of unphysical oscillations in high Pe number case introduces the formulation of Petrov-Galerkin finite element method with upwinding effect.



Amplitude ratio dependence on L/h for consistent and lumped Crank-Nicolson type schemes (From Gray and Pinder [1976])

Extension to multidimensional problems

- Splitting technique (method of fractional step)
- Alternating direction techniques
  (i) standard alternating direction technique
  (ii) tensor product alternating direction technique
  (iii) collocation alternating direction technique

Time dependent convection-diffusion problem

Consider

$$\partial\phi/\partial t + \underline{u}\cdot\nabla\phi = \nabla\cdot(k\,\nabla\phi) \quad \text{in } \Omega \qquad (k = k(\underline{x})) \quad (1)$$

$$\partial\phi/\partial n + a\,\phi = \gamma \quad \text{on } \partial\Omega \qquad (a = a(\underline{x}),\ \gamma = \gamma(\underline{x})) \quad (2)$$

$$\phi(\underline{x}, 0) = f(\underline{x}) \quad \text{in } \Omega \quad (3)$$

The weak statement for (1)-(3) is

$$\int_\Omega w\left[\partial\phi/\partial t + \underline{u}\cdot\nabla\phi - \nabla\cdot(k\nabla\phi)\right] d\Omega = 0$$

$$\rightarrow \int_\Omega w\left(\partial\phi/\partial t + \underline{u}\cdot\nabla\phi\right) + k\nabla\phi\cdot\nabla w \, d\Omega - \int_{\partial\Omega} k(\gamma - a\phi)\,w\,ds = 0 \quad (4)$$

Note — if the essential condition $\phi = g$ is proscribed on $\partial\Omega$, then the variation $w = 0$ on $\partial\Omega$ for all t and the boundary integral $= 0$.

Semi discrete finite element method implies

$$\phi(\underline{x}, t) = \sum_{j=1}^{n} \phi_j(t)\,\varphi_j(\underline{x}) \quad (5)$$

Substitute (5) into (4) → Semidiscrete Galerkin system

$$\sum_{j=1}^{N}\left[\left(\int_\Omega \varphi_i\varphi_j\,dx\right)\frac{d\phi_j}{dt} + \left(\int_\Omega (\underline{u}\cdot\nabla\varphi_j)\varphi_i\,dx\right)\phi_j \right. \quad (6)$$

$$\left. + \left(\int_\Omega k\nabla\varphi_j\cdot\nabla\varphi_i\,dx\right)\phi_j + \left(\int_{\partial\Omega} k\,a\,\varphi_i\varphi_j\,ds\right)\phi_j\right] = \int_{\partial\Omega} k\gamma\varphi_i\,d$$

for $i = 1, 2, \cdots, N$

where $w = \{\varphi_i\}$, $i = 1, 2, \cdots, N$

a system of O.D.E

$$\underline{\underline{B}}\,\frac{d\underline{\phi}}{dt} + \underline{\underline{A}}\,\underline{\phi} = \underline{c} \quad (7)$$

The initial condition is set in the form

$$\int_\Omega \phi(\underline{x},0)\, w(\underline{x})\, d\underline{x} = \int_\Omega \phi_0(\underline{x})\, w(\underline{x})\, d\underline{x}$$

or $$\underline{\underline{B}}\, \underline{\phi} = \underline{\phi}_0 \qquad \text{—(8)}$$

Time discretization for (7):

Evaluating $d\underline{\phi}/dt$ forward from t, centrally at $t+\Delta t/2$, and backward from $t+\Delta t$ respectively

$$\rightarrow \begin{cases} \underline{\underline{B}}\left(\dfrac{\underline{\phi}(t+\Delta t)-\underline{\phi}(t)}{\Delta t}\right) + \underline{\underline{A}}\,\underline{\phi}(t) = \underline{c}(t) + O(\Delta t) \\ \underline{\underline{B}}\left(\dfrac{\underline{\phi}(t+\Delta t)-\underline{\phi}(t)}{\Delta t}\right) + \underline{\underline{A}}\left(\dfrac{\underline{\phi}(t+\Delta t)+\underline{\phi}(t)}{2}\right) = \underline{c}(t+\Delta t/2) + O(\Delta t) \\ \underline{\underline{B}}\left(\dfrac{\underline{\phi}(t+\Delta t)-\underline{\phi}(t)}{\Delta t}\right) + \underline{\underline{A}}\,\underline{\phi}(t+\Delta t) = \underline{c}(t+\Delta t) + O(\Delta t) \end{cases}$$

or

$$\begin{cases} \underline{\underline{B}}\,\underline{\phi}_{j+1} = (\underline{\underline{B}} - \Delta t\,\underline{\underline{A}})\,\underline{\phi}_j + \Delta t \cdot \underline{c}_j & (9.1) \\ (\underline{\underline{B}} + \frac{\Delta t}{2}\underline{\underline{A}})\,\underline{\phi}_{j+1} = (\underline{\underline{B}} - \frac{\Delta t}{2}\underline{\underline{A}})\,\underline{\phi}_j + \Delta t \cdot \underline{c}_{j+1/2} \\ (\underline{\underline{B}} + \Delta t\,\underline{\underline{A}})\,\underline{\phi}_{j+1} = \underline{\underline{B}}\,\underline{\phi}_j + \Delta t \cdot \underline{c}_{j+1} \end{cases} \qquad \text{—(9)}$$

where $\underline{\phi}_j = \underline{\phi}(t_j)$ etc.

(9) can be solved by numerical method for O.D.E such as Runge-Kutta method.

Stability analysis:

Consider a homogeneous difference equation ($\underline{c}=0$) and examine the propagation and growth of the initial disturbation ϵ_0 by the scheme (9.1)

The error ϵ_{j+1} at time step $j+1$ satisfies

$$(\underline{\underline{I}} - \Delta t\, \underline{\underline{B}}^{-1}\underline{\underline{A}})^{-1}\underline{\epsilon}_{j+1} = \underline{\epsilon}_j \qquad \text{— (10)}$$

Thus the generalized eigenvalue problem is important in analyzing the stability.

Let $\{\lambda_i\}$, $\{v_i\}$ be the eigenvalues and eigenvectors of $\underline{\underline{A}}\,\underline{v} = \lambda\, \underline{\underline{B}}\,\underline{v}$ where $\underline{\underline{A}}, \underline{\underline{B}}$ are $N \times N$ matrices.

On repeated application of (10)

$$\rightarrow \quad \underline{\epsilon}_{j+1} = (\underline{\underline{I}} - \Delta t\, \underline{\underline{B}}^{-1}\underline{\underline{A}})^{j+1}\underline{\epsilon}_0 \qquad \text{— (11)}$$

Expanding the initial error vector $\underline{\epsilon}_0$ in eigenvectors

$$\underline{\epsilon}_0 = \sum_{i=1}^{N} \bar{\alpha}_i\, \underline{v}_i$$

$$\begin{aligned}
\rightarrow \quad \underline{\epsilon}_{j+1} &= (\underline{\underline{I}} - \Delta t\, \underline{\underline{B}}^{-1}\underline{\underline{A}})^{j}(\underline{\underline{I}} - \Delta t\, \underline{\underline{B}}^{-1}\underline{\underline{A}})\,\underline{\epsilon}_0 \\
&= (\underline{\underline{I}} - \Delta t\, \underline{\underline{B}}^{-1}\underline{\underline{A}})^{j}\left(\sum_{i=1}^{N} \bar{\alpha}_i\, \underline{v}_i - \Delta t\, \underline{\underline{B}}^{-1}\underline{\underline{A}}\sum_{i=1}^{N}\bar{\alpha}_i\, \underline{v}_i\right) \\
&= (\underline{\underline{I}} - \Delta t\, \underline{\underline{B}}^{-1}\underline{\underline{A}})^{j}\sum_{i=1}^{N}(1-\lambda_i \Delta t)\,\bar{\alpha}_i\, \underline{v}_i \\
&= \sum_{i=1}^{N}(1-\lambda_i\Delta t)^{j+1}\,\bar{\alpha}_i\, \underline{v}_i
\end{aligned}$$

$\Rightarrow$ for each eigenvector $\underline{v}_i$, $(1-\lambda_i \Delta t)^{j+1}$ is the amplification factor of $\underline{\epsilon}_0$

if $|1-\lambda_i \Delta t| < 1$

$\rightarrow$ no component of the error will grow and the method is stable.

Lumping approximation :

The coefficient matrix $\underline{\underline{B}}$ of the time derivative term in

$$\underline{\underline{B}}\ d\phi/dt + \underline{\underline{A}}\ \underline{\phi} = \underline{c}$$

is not diagonal.

Lumping process results in a diagonal matrix $\underline{\underline{B}}$.

This idea stems from engineering analysis of vibration problems in which the model is simplified by replacing a continuous mass distribution by a system of point masses lumped at the nodes.

$$\underline{\underline{B}}^e = b^e_{ij} = \int_{\Omega^e} \psi_i \psi_j \, dx = \int_{\hat{\Omega}} \hat{\chi}_i \hat{\chi}_j \, |J| \, d\xi$$

Where $\{ \chi_i \}$, $i=1,2$ are local basis functions for element Ω_e. and $\hat{\Omega} = [-1, 1]$.

$$b^e_{ij} \sim \frac{h_e}{2} \sum_{k=1}^{2} w_k \hat{\chi}_i(\xi_k) \hat{\chi}_j(\xi_k)$$

Where $\xi_k = \pm 1$, and quadrature points $w_k = 1/2$

$$\rightarrow \underline{b}^e \sim \frac{h_e}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (12)$$

For the model 1-D problem $\partial\phi/\partial t + \alpha\, \partial\phi/\partial x = \beta^2\, \partial^2\phi/\partial x^2$

one can combine element contributions at interior node i in a uniform mesh h to have lumped discretization equation

$$\frac{d\phi_i}{dt} = \left(\frac{\beta^2}{h^2} + \frac{\alpha}{2h}\right)\phi_{i-1} - \frac{2\beta^2}{h^2}\phi_i + \left(\frac{\beta^2}{h^2} - \frac{\alpha}{2h}\right)\phi_{i+1} \qquad (13)$$

Forward diff on (13) results in

$$\phi_i^{j+1} = \phi_i^j + \Delta t \left\{ \left(\beta^2/h^2 + \alpha/2h\right)\phi^j_{i-1} - 2\beta^2/h^2\, \phi^j_i + \left(\beta^2/h^2 - \alpha/2h\right)\phi^j_{i+1} \right\}$$

or $\underline{\phi}^{\nu+1} = (\underline{\underline{I}} - \Delta t\ \underline{\underline{D}}^{-1} \underline{\underline{A}})\ \underline{\phi}^{\nu}$ —— (14)

<u>Note</u> : The corresponding consistent discretization scheme for (12) is

$$\frac{h}{6}\left(\frac{d\phi_{i-1}}{dt} + 4\ \frac{d\phi_i}{dt} + \frac{d\phi_{i+1}}{dt} \right) + \frac{\alpha}{2}(\phi_{i+1} - \phi_{i-1})$$

$$- \frac{\beta^2}{h}(\phi_{i+1} - 2\phi_i + \phi_{i-1}) = 0 \qquad \text{—— (15)}$$

The stability condition for (14) is then

$| 1 - \Delta t\ \lambda_j | < 1$

where λ_j is the eigenvalues of $\underline{\underline{D}}^{-1} \underline{\underline{A}}$.

<i> $\beta^2/h^2 \geq \alpha/2h$ (or $\alpha h/\beta^2 < 2$, i.e. Peclet no ≤ 2)

$\lambda_i = \frac{2\beta^2}{h^2} - 2\left\{ (\beta^2/h^2 + \alpha/2h)(\beta^2/h^2 - \alpha/2h) \right\}^{1/2} \cos \frac{i\pi}{N+1}$, $i = 1, 2, \cdots, N$

$\rightarrow \quad -1 < 1 - \Delta t\ \mu < 1 \qquad$ (if $\beta^2/h^2 \geq \alpha/2h$)

where $\mu = Re(\lambda_i)$

$\Longrightarrow \quad \Delta t < h^2/2\beta^2$

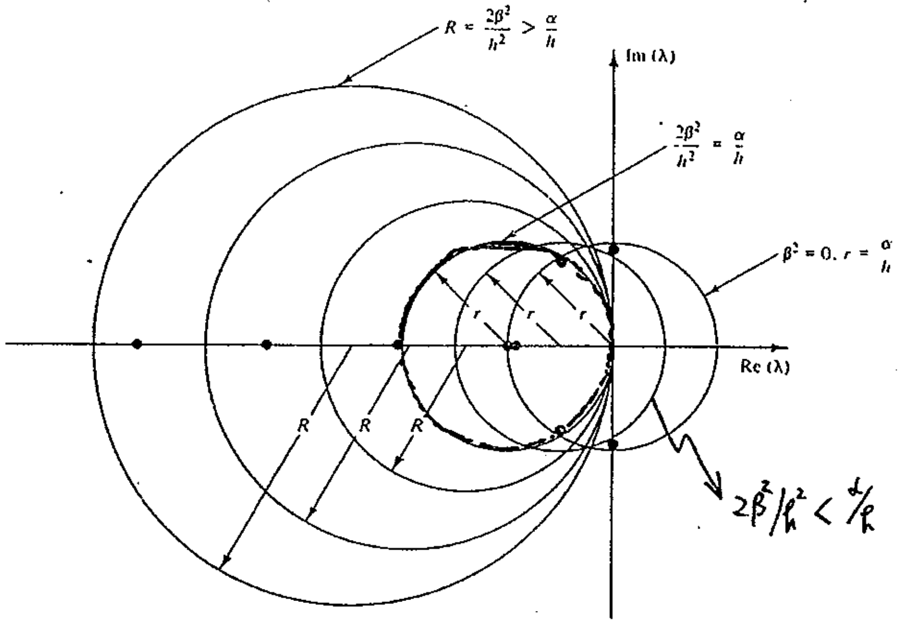
<ii> if $\beta^2/h^2 < \alpha/2h$ (or Peclet no $\alpha h/\beta^2 > 2$).

$\Delta t \leq 4\beta^2/\alpha^2$ the eigenvalues λ_j are complex.

<u>Note</u> : The eigenvalues λ_i of $\underline{\underline{B}}^{-1} \underline{\underline{A}}$ is complex if convection dominates. The imaginary parts of these eigenvalues play an important role in explaining the presence of oscillatory ripples.

[∵ if the eigenvalues are complex, each component in the eigenvector expansion of the initial vector has an associated amplitude and phase → the inability of a given numerical scheme to propagate individual frequencies without phase errors and with accurate amplitudes leads to oscillations]

- As $\beta^2 = 0$, the circles G_i are centered at origin and have radius α/h. As $\alpha \uparrow$ or $h \downarrow$, the circles grow.
- Introducing a small amount of diffusion but keeping $2\beta^2/h^2 < \alpha/h \rightarrow r_i = \alpha/h$ and origin centered at $-2\beta^2/h^2$
  $\longrightarrow$ the imaginary axis intersects the circle and the system is still oscillatory.
- As $2\beta^2/h^2 = \alpha/h \rightarrow$ convection and diffusion balance
- As β^2 continues to increase $\rightarrow$ diffusion dominates and the circles are all tangent to the imaginary axis.



- Keeping α, β^2 fixed, one can simply change situation $2\beta^2/h^2 < \alpha/h$ to $2\beta^2/h^2 > \alpha/h$ by decreasing mesh size h. $\rightarrow$ refinement enhances stability.
- The lumped upwind scheme for interior node i is

$$\frac{d\phi_i}{dt} = \left(\frac{\beta^2}{h^2} + \frac{\alpha}{2h} + \frac{\omega\alpha}{6h}\right)\phi_{i-1} - \left(\frac{2\beta^2}{h^2} + \frac{\omega\alpha}{3h}\right)\phi_i + \left(\frac{\beta^2}{h^2} - \frac{\alpha}{2h} + \frac{\omega\alpha}{6h}\right)\phi_{i+1}$$

Some insights on model convection diffusion problem:

Consider model problem $\phi_t + \alpha \phi_x = \beta^2 \phi_{xx}$

The associated lumped system can be rewritten as

$$\frac{d\underline{\phi}}{dt} = \left(-\alpha \underline{\underline{A}}_1 + \beta^2 \underline{\underline{A}}_2\right) \underline{\phi} \qquad \text{— (1)}$$

Using a piecewise-linear f.e basis function in uniform mesh one can derive a equation for node i

$$\frac{d\phi_i}{dt} = \left(\frac{\beta^2}{h^2} + \frac{\alpha}{2h}\right)\phi_{i-1} - \frac{2\beta^2}{h^2}\phi_i + \left(\frac{\beta^2}{h^2} - \frac{\alpha}{2h}\right)\phi_{i+1} \qquad \text{—(2)}$$

Gershgorin circle theorem —

The eigenvalues $\{\lambda_i\}$ of matrix $\underline{A} = (a_{ij})$ lie in the union of the Gershgorin disks G_i defined by

$$|\lambda - a_{ii}| \le \sum_{\substack{j=1 \\ j\ne i}}^{n} |a_{ij}| \qquad \text{complex } \lambda$$

ie, the disks G are centered in the complex λ plane at points a_{ii} and its radius is equal to the sum of the off-diagonal entries for $i = 1, 2, \dots, n$

- This theorem provides a simple means of computing bounds on the eigenvalues and can be applied to study the effects of mesh variation in a quantative manner

From (2) one can obtain the Gershgorin bound

$$\left|\lambda - \left(-\frac{2\beta^2}{h^2}\right)\right| \le \left|\frac{\alpha}{2h} + \frac{\beta^2}{h^2}\right| + \left|-\frac{\alpha}{2h} + \frac{\beta^2}{h^2}\right|$$

ie. G_i are centered at $-2\beta^2/h^2$ and have radius

$$r_i = \left|\beta^2/h^2 + \alpha/2h\right| + \left|\beta^2/h^2 - \alpha/2h\right| = \begin{cases} 2\beta^2/h^2 & \text{if } \beta^2/h^2 > \alpha/2h \\ \alpha/h & \text{if } \beta^2/h^2 < \alpha/2h \end{cases}$$

(d) $\frac{d\phi_i}{dt} = \left(\frac{\beta^2}{h^2}+\frac{\alpha}{2h}\right)\phi_{i-1} + \left(-\frac{2\beta^2}{h}\right)\phi_i + \left(\frac{\beta^2}{h^2}-\frac{\alpha}{2h}\right)\phi_{i+1}$

$$+\omega\left[\left(\frac{\alpha}{6h}\right)\phi_{i-1} + \left(-\frac{\alpha}{3h}\right)\phi_i + \left(\frac{\alpha}{6h}\right)\phi_{i+1}\right]$$

where ω is the upwinding parameter.

One can obtain the Gershgorin bound

$$\left|\lambda - \left(-\frac{2\beta^2}{h^2} - \frac{\omega\alpha}{3h}\right)\right| \le \left|\frac{\beta^2}{h^2}+\frac{\alpha}{2h}+\frac{\omega\alpha}{6h}\right| + \left|\frac{\beta^2}{h^2}-\frac{\alpha}{2h}+\frac{\omega\alpha}{6h}\right|$$

(e), the circles are centered at $-2\beta^2/h^2 - \omega\alpha/3h$ and the radius

$$r_i = \begin{cases} \alpha/h & \text{if } 2\beta^2/h^2 + \omega\alpha/3h < \alpha/h \text{ (convect dominate)} \\ 2\beta^2/h^2 + \omega\alpha/3h & \text{if } 2\beta^2/h^2 + \omega\alpha/3h \ge \alpha/h \end{cases}$$

→ ‹i› the upwinding shift the circle center from $-2\beta^2/h^2$ to the left at $-2\beta^2/h^2 - \omega\alpha/3h$.

‹ii› upwinding in convection dominated situations reduces the imaginary parts of the eigenvalues and thus oscillations.

Stability analysis

Consider a parabolic equation which can be discretized by various procedure to

$$\underline{\underline{B}}\, {d\underline{\phi}}/{dt} + \underline{\underline{A}}\,\underline{\phi} = \underline{f}$$

or

$$ {d\underline{\phi}}/{dt} = -\,\underline{\underline{B}}^{-1}\underline{\underline{A}}\,\underline{\phi} + \underline{\underline{B}}^{-1}\underline{f} \qquad \text{—— (1)}$$

where $\underline{\phi}(t)$ is nodal solution vector

$\underline{\phi}(0) = \underline{\phi}_0$ is initial data

The exact solution for (1) is

$$\underline{\phi}(t) = \underline{\underline{A}}^{-1}\underline{\underline{B}}\,\underline{f} + \exp(-t\,\underline{\underline{B}}^{-1}\underline{\underline{A}})(\underline{\phi}_0 - \underline{\underline{A}}^{-1}\underline{\underline{B}}\,\underline{f})$$

Note : if $\underline{f} = 0$,

define $\underline{\underline{M}} = \underline{\underline{B}}^{-1}\underline{\underline{A}}$

$$\rightarrow \quad \underline{\phi}(t) = \exp(-t\,\underline{\underline{M}})\,\underline{\phi}_0$$

Note :
$$\exp(-t\,\underline{\underline{M}}) = \sum_{j=0}^{\infty} \frac{1}{j!}(-t\,\underline{\underline{M}})^j$$
$$\approx \underline{\underline{I}} - t\,\underline{\underline{M}} + \tfrac{1}{2}(t\,\underline{\underline{M}})^2 + o(t^3)$$
$$\simeq \underline{\underline{I}} - t\,\underline{\underline{M}} + o(t^2)$$

Let $\underline{\phi}_0 = \sum_{j=1}^{n} \alpha_j \underline{v}_j$. **where** $\underline{v}_j$ are eigenvectors

$$\rightarrow \quad t\,\underline{\underline{M}}\,\underline{\phi}_0 = t \sum_{j=1}^{n} \alpha_j\,\underline{\underline{M}}\,\underline{v}_j = t \sum_{j=1}^{n} \alpha_j \overbrace{\lambda_j(\underline{\underline{M}})}^{\text{eigenvalue of } \underline{\underline{M}}}\,\underline{v}_j$$

$$\rightarrow \quad \underline{\phi}(t) \approx (\underline{\underline{I}} - t\,\underline{\underline{M}})\,\underline{\phi}_0$$
$$\approx \underline{\phi}_0 - t \sum_{j=1}^{n} \alpha_j \lambda_j(\underline{\underline{M}})\,\underline{v}_j$$

$$\underline{\Phi}(t) \approx \sum_{j=1} \alpha_j \underline{v}_j - t \sum_{j=1} \alpha_j \lambda_j(\underline{\underline{M}}) \underline{v}_j$$

$$= \sum_{j=1}^{n} \alpha_j \underline{v}_j \left(1 - t \lambda_j(\underline{\underline{M}}) \right)$$

It implies that (i) the ratio of decay of a component is determined by the magnitude of the real part of the eigenvalues $\lambda_j(\underline{\underline{M}})$

(ii) oscillatory behavior (phase) of each component depends on the relative magnitude of the imaginary parts of $\lambda_j(\underline{\underline{M}})$

Gershgorin theory

Theorem:

The eigenvalues $\{\lambda_k\}$ of a matrix $\underline{\underline{M}}$ lie in the union of the Gershgorin discs G_i, defined by

$$d\underline{\phi}/dt = -\underline{\underline{M}}\,\underline{\phi}$$

$$|\lambda - m_{ii}| \le \sum_{j\ne i}^{n} |m_{ij}| \quad , \quad \text{complex } \lambda$$

Oscillating matrix

Definition:

totally non-negative (totally positive) matrix $\underline{\underline{A}}$

— if $\underline{\underline{A}} = (a_{ij})$, if all its minors of any order are non-negative (positive)

Definition: Oscillatory matrix

— A matrix $\underline{\underline{A}}$ is oscillatory if $\underline{\underline{A}}$ is non-negative and there exists an integer $p>0$ such that $\underline{\underline{A}}^p$ is totally positive

Theorem:

A ~~totally~~ non-negative matrix $\underline{\underline{A}}$ is oscillatory iff

(i) $\underline{\underline{A}}$ is non-singular

(ii) all elements of $\underline{\underline{A}}$ in the main diagonal and first sub- and sup-diagonals are positive,

i.e., $a_{ij} > 0$ for $|i-j| \le 1$, $i,j = 1,2,\cdots,n$

<u>Example:</u> consider linear C-D equation

$$\phi_t + \alpha \phi_x = \beta^2 \phi_{xx}$$

where α, β^2 are constants.

If a representative equation for i is (central-based scheme)

$$d\phi_i/dt = \left(\beta^2/h^2 + \alpha/2h\right)\phi_{i-1} - \frac{2\beta^2}{h^2}\phi_i + \left(\beta^2/h^2 - \alpha/2h\right)\phi_{i+1} \qquad (2)$$

where h is uniform space interval.

Apply Gershgorin theorem; the disk ϕ_i is

$$\rightarrow \quad \left|\lambda - \left(-\frac{2\beta^2}{h^2}\right)\right| \le \left|\frac{\alpha}{2h} + \frac{\beta^2}{h^2}\right| + \left|-\frac{\alpha}{2h} + \frac{\beta^2}{h^2}\right|$$

ie, the disc is centered at $-\frac{2\beta^2}{h^2}$ but of radius

$$r_i = \begin{cases} 2\beta^2/h^2 & \text{if } \beta^2/h^2 > \alpha/2h \quad (\text{diffusion dominated}) \\ \alpha/h & \text{if } \beta^2/h^2 < \alpha/2h \quad (\text{convection dominated}) \end{cases}$$

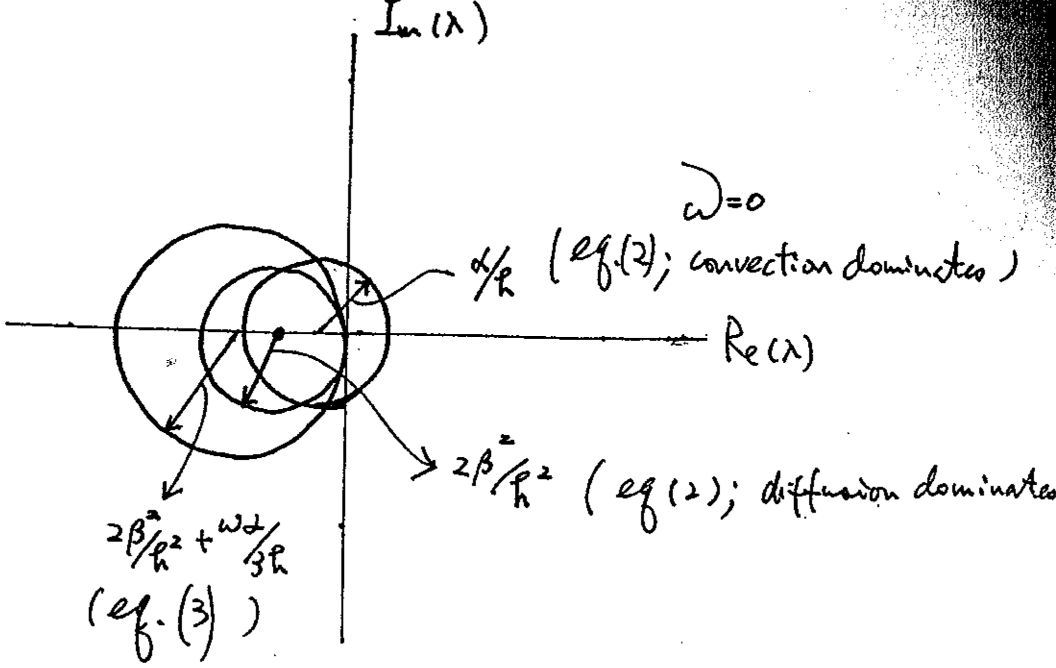
Suppose (2) is re-structured to upwind type by introducing an upwind parameter ω, for example

$$\rightarrow \quad d\phi_i/dt = \left(\beta^2/h^2 + \alpha/2h + \omega\alpha/6h\right)\phi_{i-1}$$
$$- \left(2\beta^2/h^2 + \omega\alpha/3h\right)\phi_i + \left(\beta^2/h^2 - \alpha/2h + \omega\alpha/6h\right)\phi_{i+1} \qquad (3)$$

The circle of disc is $-2\beta^2/h^2 \boxed{-\omega\alpha/3h}$ of radius

$$r_i = \begin{cases} \alpha/h & ; \text{ if } 2\beta^2/h^2 + \omega\alpha/3h < \alpha/h \\ 2\beta^2/h^2 + \omega\alpha/3h & ; \text{ if } 2\beta^2/h^2 + \omega\alpha/3h \geq \alpha/h \end{cases}$$

ie, * the effect of upwinding is to translate the circle center to the left. $\longrightarrow$ reducing the imaginary part of the eigenvalues and mitigates oscillations



Note: if ω keeps increasing, the system becomes over-dissipative and lower the ability to resolve sharp fronts.

Note: this analysis ~~can~~ provide a direct geometric interpretation of the investigated methods in the area of performance study.

Convective transport problems

* Conventional Galerkin finite element methods lead to central difference approximation
  ⇒ Like standard finite difference method, it produces spurious node-to-node oscillations or wiggles for convective transport equations. It may blur out completely the solution of the problem.

To preclude such oscillations
→ (1) finite difference methods:
  → Performing upwind differencing on the convective terms. At the expense of one-order of convergence rate
  (2) finite element methods:
  → Modified weighting functions are employed to achieve the desired effects

(A generalized Galerkin method is analogous to upwind scheme of finite difference method) ⇒ Consistent Petrov-Galerkin weighted residual method is defined as the modified weighting function is applied to all terms in the equation.

I. Christie, D.F. Griffiths, A.R. Mitchell, O.C. Zienkiewicz (1976)

Techniques to construct Petrov-Galerkin formulations

(1) Morton and Parrott (1980):
special test functions are developed so that signals are propagated without distortion when characteristics pass through mesh points at each time level.

(2) Brooks and Hughes (1981):
Construct upwind finite elements via Galerkin method with added artificial diffusion.

or Petrov Galerkin finite element method.

Modified weighted residual (MWR)

(A) For constant coefficient case:

Considering one dimensional stationary heat equation.

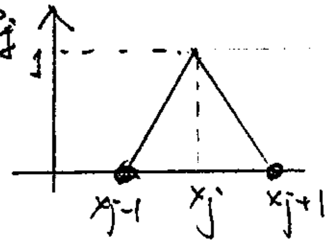
$$\begin{cases} u T_x - K T_{xx} = 1 \\ T(0) = T(1) = 0 \end{cases}$$

Standard weighted residual method requires that the residual $(uT_x - KT_{xx} - 1)$ should be orthogonal to all function ϕ over the space, i.e,

$$\int_\Omega \phi_j (u T_x - K T_{xx} - 1)\, dV = 0$$

where the dependent variable T is approximated by shape function ϕ_i

$$T(x) = \sum \phi_i(x) T_i$$

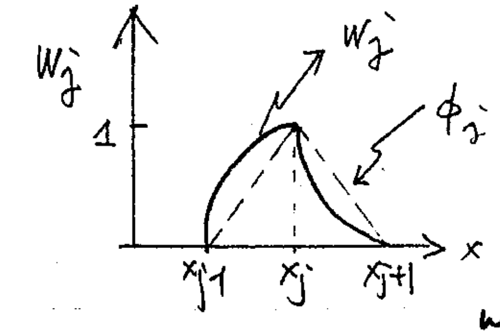


If the weighting functions W_j are different from shape functions ϕ_j

$$W_j(x) = \begin{cases} \phi_j(x) + \dfrac{\alpha_j}{h_j} N\left(\dfrac{x - x_{j-1}}{h_j}\right) & \text{if } x_{j-1} \le x \le x_j \\ \phi_j(x) - \dfrac{\alpha_j}{h_{j+1}} N\left(\dfrac{x_{j+1} - x}{h_{j+1}}\right) & \text{if } x_j \le x \le x_{j+1} \end{cases}$$

by adding a quadratic which vanishs at the nodes

where $N(\xi) = 3\xi(1-\xi)$

$\rightarrow$ 

weighting functions w_j are sucked back in the upstream direction.

upwind-biased test function

Evaluating

$$\int_x w_j \left(u \sum_{i=1}^{2} \phi_{i,x} T_i - K \sum_{i=1}^{2} \phi_{i,xx} T_i - 1 \right) dx = 0$$

$$\rightarrow \quad K \left(1 + \frac{u\alpha}{2K} \Delta x \right) \frac{2T_j - T_{j-1} - T_{j+1}}{(\Delta x)^2} + u \frac{T_{j+1} - T_{j-1}}{2(\Delta x)} = 1$$

where constant u and uniform spacing are assumed. (16)

Comparing the above relation with the exact finite difference expression

$$u \frac{T_{j+1} - T_{j-1}}{2\Delta x} + K \frac{r}{2} \coth\left(\frac{r}{2}\right) \frac{T_{j+1} - 2T_j + T_{j-1}}{(\Delta x)^2} = 1$$

where $r = \frac{u \Delta x}{K}$ = cell Reynold no. (17)

one can identify the numerical dissipation

$$K \left(1 + \frac{u\alpha}{2K} \Delta x \right) - K \frac{r}{2} \coth \frac{r}{2}$$

$\Rightarrow$ The optimal choice for α is then

$$1 + \frac{u\alpha}{2K} \Delta x = + \frac{r}{2} \coth \frac{r}{2}$$

or $\boxed{\alpha = \coth\left(\frac{r}{2}\right) - \frac{2}{r}}$ is the optimal value.

Comparing (16) with the central difference form

$$K\frac{2T_j - T_{j-1} - T_{j+1}}{(\Delta x)^2} + u\frac{T_{j+1}-T_{j-1}}{2(\Delta x)} = 1$$

(which can also be obtained by replacing the weighting functions W_j by shape functions ϕ_j.) Homework

→ The numerical viscosity is added by an amount $K' = K\frac{\alpha \gamma}{2}$ due to the use of $\underbrace{\text{weighting functions } W_j}_{\text{additional quadratic in the}}$.

- For large γ:
  The optimal $\alpha = 1 - 2/\gamma$ is employed.
- Straighforwardly extended to 2-D for quadrangles.
- effective but increased computational complexity.
- Based on the knowledge that

  $\frac{u\Delta x}{K} \equiv \gamma \geq 2 \rightarrow$ wiggles will occur in the centered difference scheme

  $$K\frac{2T_j - T_{j-1} - T_{j+1}}{(\Delta x)^2} + u\frac{T_{j+1}-T_{j-1}}{2\Delta x} = 1$$

  One can have the conclusion from equation (16) that

  $$\frac{u\Delta x}{K(1+\frac{u\alpha}{2K}\Delta x)} \leq 2 \quad \text{critical value.}$$

  or $\alpha \geq 1 - 2/\gamma$ where $\gamma = \frac{u\Delta x}{K}$

(B) Variable upwinding and adaptive mesh refinement for convection-diffusion equations.

- Particularly well suitable to problems with nonconstant coefficients, nonlinearity or non-uniform meshes.
- Upwinding corresponds to adding dissipation which may produce over-dissipative solutions
  smearing out
- Standard Galerkin finite element methods have oscillations if convection is large and elements are not sufficiently small.

$\longrightarrow$ It suggests that mesh refinement may be a preferable alternative to upwinding since a local cell Reynold number yields a bound on the acceptable mesh size.

$\longrightarrow$ Adaptive refinement is required.

(a) convection-diffusion model problem

$$u_t + \alpha u_x - (\beta^2 u_x)_x = 0 \qquad (1)$$

where $\alpha = \alpha(x,t)$ and $\beta = \beta(x,t)$

$0 < x < 1, \quad 0 \le t \le T.$

B.C: $$u(0,t) = \gamma_0(t)$$ $$u(1,t) = \gamma_1(t) \qquad (2)$$

I.C: $$u(x,0) = u_0(x) \qquad (3)$$

and $\alpha,\ \beta^2$ are constants

Since $\int v \beta^2 u_{xx}\, dx$

$$= \int \beta^2 [(v u_x)_x - v_x u_x] dx$$

$$= -\int \beta^2 v_x u_x\, dx + \int \beta^2 (v u_x)_x\, dx$$

then $\int_0^1 v \left[u_t + \alpha u_x - (\beta^2 u_x)_x \right] dx$

$$= \int_0^1 (v u_t) + u_x v \alpha + \beta^2 v_x u_x\, dx - \int \beta^2 (v u_x)_x\, dx$$

$$= \int_0^1 v u_t\, dx + \int_0^1 \alpha u_x v\, dx + \int_0^1 \beta^2 u_x v_x\, dx = 0 \qquad (4)$$

where $v \in H_0^1(0,1)$, the Hilbert space of test functions which vanish at the end boundary points and have ~~square~~ integrable derivatives.

Choosing approximate solution of u by $u_h(x,t)$

$$u_h(x,t) = \sum_{j=1}^{N} u_j(t)\, \phi_j(x) \qquad (5)$$

where $\{\phi_j\}$ be a piecewise smooth basis functions.

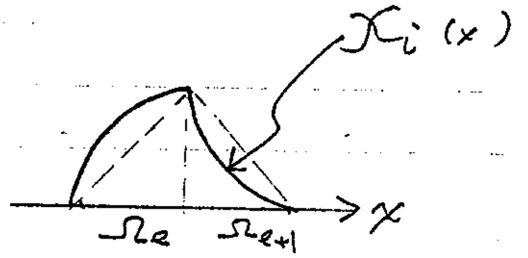
Substitute (5) into weak statement (4) where v is replaced by $\underbrace{\{\chi_i\}}_{\text{test space}}$

$$\rightarrow \sum_{j=1}^{N} \left\{ \left(\int_0^1 \chi_i \phi_j\, dx \right) \frac{du_j}{dt} + \left(\int_0^1 \alpha \chi_i \phi_j'\, dx \right) u_j + \left(\int_0^1 \beta^2 \chi_i' \phi_j'\, dx \right) u_j \right\} = 0$$

where $i = 2, 3, \cdots, N-1$ since $u(0,t)$, $u(1,t)$ are known or $\underline{B}\frac{du}{dt} + \underline{A}\,\underline{u} = \underline{f}$

- The test functions $\{\chi_i\}$ are piecewise linear with an added quadratic upwind bias of strength ω

ie, $\chi_i = \phi_i + \omega g = \phi_i + \omega l_1 l_2$



- The presence of banded $\underline{B}$ has been recognized as possible disvantage in f.e.m. of transient problem,

$\longrightarrow$ lumped finite element methods are employed
ie, the test functions in first term are different from other
or the first term is under-integrated employing Newton-Cotes quadrature on each element.

$\Longrightarrow$ The approximated matrix of $\underline{B}$ becomes diagonal matrix D

or $du/dt = -\hat{A}\,\underline{u} + \hat{\underline{f}}$

where $\hat{A} = \hat{A}_{ij} = h^{-1} A_{ij}$,
$\hat{f} = f_i / h$ $\qquad h = D_{ii}$

- A representative equation for node i for lumped linear elements is

$$\frac{du_i}{dt} = \left(\frac{\beta^2}{h^2} + \frac{\alpha}{2h} + \frac{\omega\alpha}{6h} \right) u_{i-1} - \left(\frac{2\beta^2}{h^2} + \frac{\omega\alpha}{3h} \right) u_i + \left(\frac{\beta^2}{h^2} - \frac{\alpha}{2h} + \frac{\omega\alpha}{6h} \right) u_{i+1}$$

→ Gerchgorin theorem, the optimal choice for ω^* to ensure the diffusion and convection balance

$$\omega^* = 3 - \frac{6\beta^2}{\alpha h} = 3 - \frac{6}{P_e} \quad \text{where } P_e = \frac{\alpha h}{\beta^2}$$

ie, if $P_e < 2$ no upwind is necessary

→ uniform refinement may produce non-oscillatory solutions.

- Uniform refinement techniques are not appropriate since the problem size rapidly becomes too large for large P_e flows.

→ Adaptive local refinement. which is based on the element residuals.

ie, the solution is computed on an initial coarse mesh → areas where element residuals are large are identified → the mesh is refined locally → the solution is then obtained on the new mesh.

⇛ The effect of upwinding and of mesh refinement is to generate oscillations, it follows that if uniform or local refinement is to be utilized in conjunction with upwinding, then ω should be variable, ie, $\omega = \omega(h)$.

Note:

Petrov - Galerkin method — unlike Galerkin method, the weighting functions and trial functions differ. This method can remove wiggles in the velocity vector field using Galerkin method.

Int. J. Num. Mech. Eng. 12, (1978) p. 1359-1365.

Hughes 1978, 1979,
Raviart 1979,
Dervieux 1979.

Reduced integration of advection

Consider the equation (one-dimensional case)

$$\left\{\begin{array}{l} \underline{u}\cdot\underline{\nabla}T - K\nabla^2 T = f \\ \text{with B.C.} \\ T|_\Gamma = 0 \end{array}\right. \qquad (1)$$

The finite element equations of (1) can be written as

$$\underline{A}\,\underline{X} = \underline{F}$$

where $\underline{X} = \begin{pmatrix} T_1 \\ \vdots \\ T_N \end{pmatrix}$; $\underline{F} = \begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix}$; $\underline{A} = [a_{ij}]$

$$f_i = \int_\Omega f\,\phi_j\,dx$$

$$a_{ij} = \sum_K a_{ij}^K = \sum_K \int_K \phi_j\,(\underline{u}\cdot\underline{\nabla}\phi_i - K\nabla^2\phi_i)\,dx$$

$$= \sum_K \int_K \phi_j\,(\underline{u}\cdot\nabla\phi_i) + K\,\underline{\nabla}\phi_i\cdot\underline{\nabla}\phi_j\,dx \qquad (2)$$

Since only volume terms are existed in a_{ij} and ϕ_i are the shape functions for

$$T^e(x) = \sum_i \phi_i(x)\,T_i$$

K: denoting element.

The a_{ij}^K in equation (2) is modified by

$$a_{ij}^K = \left(\int_K K\,\underline{\nabla}\phi_i\cdot\underline{\nabla}\phi_j\,dx\right)$$
$$+ \mathrm{Area}(K) \times \phi_j(x^K)\,\underline{u}(O^K)\cdot\underline{\nabla}\phi_i(x^K)$$

where O^k means the centre of element k

x_*^k means some points within the element k

the position determines the degree of upwinding

(A) For quadrangle element

(1) For one-dimensional case :

k is a typical element located between x_{j-1}, x_j with length $h_{j-1/2} = x_j - x_{j-1}$

Then

$$O^k = \frac{1}{2}(x_j + x_{j-1}) \equiv x_{j-1/2}$$

Let $x^k = x_{j-1/2} + \alpha_{j-1/2}(h_{j-1/2}/2)$

If 1-point Gauss formula for the integration is used and uniform spacings and constant u are assumed

$$\Rightarrow \frac{K}{(\Delta x)^2}(-T_{j-1} + 2T_j - T_{j+1}) + u\left(\frac{1+\alpha}{2}\right)(T_j - T_{j-1})\frac{1}{\Delta x} + u\left(\frac{1-\alpha}{2}\right)\frac{1}{\Delta x}(T_{j+1} - T_j) = f_j$$

is obtained.

or

$$\frac{K}{(\Delta x)^2}(2T_j - T_{j+1} - T_{j-1}) + \frac{u}{2\Delta x}(T_{j+1} - T_{j-1}) + u\left[\frac{\alpha}{2\Delta x}(2T_j - T_{j-1} - T_{j+1})\right] = f_j$$

or

$$(2T_j - T_{j-1} - T_{j+1})\left[\frac{K}{(\Delta x)^2} + \frac{u\alpha}{2(\Delta x)}\right] + \frac{u}{2\Delta x}(T_{j+1} - T_{j-1}) = 0$$

$$\left(K + \frac{u\alpha(\Delta x)}{2}\right)\frac{2T_j - T_{j-1} - T_{j+1}}{(\Delta x)^2} + \frac{u}{2\Delta x}(T_{j+1} - T_{j-1}) = f_j$$

$$\rightarrow \quad \frac{u\Delta x}{k + \frac{u\alpha\Delta x}{2}} \leq 2 \qquad \text{for stability}$$

$$\text{or} \quad \frac{u\,\Delta x}{k\left(1+\frac{u\alpha\Delta x}{2k}\right)} \leq 2 \qquad \gamma \equiv u\Delta x/k$$

or $\alpha \geq 1 - 2/\gamma$ critical value.

$\Rightarrow$ if $|\gamma| < 2$ then, it is stable for all α.

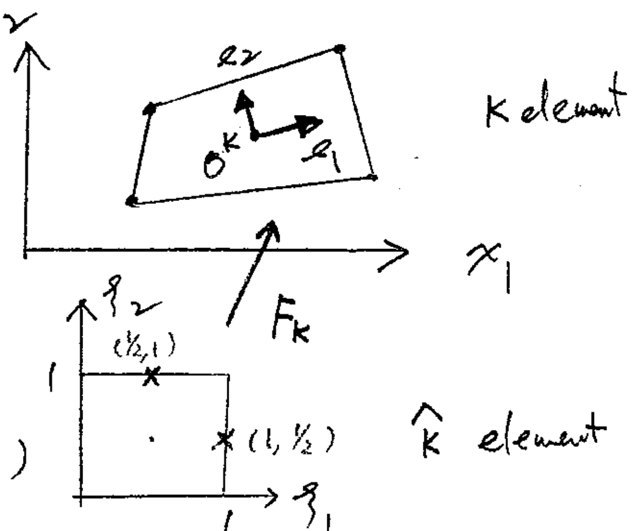
$\begin{cases} \gamma > 1 \\ \gamma < -2 \end{cases}$ then, it is stable for $\alpha > 1 - 2/\gamma$

then, it is stable for $\alpha < -1 + 2/\gamma$

$\alpha = \coth\frac{\gamma}{2} - \frac{\gamma}{2}$ gives optimal accuracy. In order to achieve the upstream discretization one has to evaluate the shape function at x^k, downstream of O^k where u is evaluated.

(ii) For 2-D case:

Bilinear isoparametric element. $\hat{K} \xrightarrow{F} K$



$e^1 = F_K(1, 0.5) - O^K \quad (\xi_1, \xi_2)$

$e^2 = F_K(0.5, 1) - O^k$

element Reynold number $\gamma_i = u(O^k)\cdot e_i / k(O^k) \quad (i = 1, 2)$

$x^k = F_K(\xi_1, \xi_2)$ where

ξ_1 ξ_2 are computed from the values of γ_1, γ_2, and the one dimensional above rule.

(B) It can be extended to triangle element.

Upwind finite element method via numerical integration. Guy Payre, U. of Montreal, 1982.

Finding the approximation to $\phi(x,y)$ satisfying

$$u\phi_x + v\phi_y - k(\phi_{xx} + \phi_{yy}) = 0 \quad \text{in } \Omega \qquad (1)$$
$$\phi = f \quad \text{on } \Gamma \qquad (2)$$

where f is a given function defined on Γ

The weak form of (1) is taken to be

$$\int_\Omega w\left[u\phi_x + v\phi_y - k(\phi_{xx} + \phi_{yy})\right] dx\,dy = 0$$

where w is test function which vanishes on Γ.

Integration by parts on diffusion terms, one obtains

$$\int_\Omega w(u\phi_x + v\phi_y) - k\left[(w\phi_x)_x - w_x\phi_x + (w\phi_y)_y - w_y\phi_y\right] dx\,dy = 0$$

or $$\int_\Omega \left[w(u\phi_x + v\phi_y) + k(w_x\phi_x + w_y\phi_y)\right] dx\,dy = 0$$

since w is equal to zero on the boundary.

If the isoparametric assumption is made,

$$\phi(x,y) = \sum_{i=1}^{N} \phi_i N_i(x,y)$$ where N: the nodal number over each element (*)

$$x = x(\xi,\eta) = \sum_{i=1}^{4} N_i(\xi,\eta)\,x_i$$
$$y = y(\xi,\eta) = \sum_{i=1}^{4} N_i(\xi,\eta)\,y_i$$

$$\Rightarrow \int_\Omega N_i (u N_{j,x} \phi_j + v N_{j,y} \phi_j) + k (N_{i,x} N_{j,x} \phi_j + N_{i,y} N_{j,y} \phi_j) dxdy = 0$$

for $i = 1, 2, \ldots, N$

or $$\left[\int_\Omega u N_i N_{j,x} + v N_i N_{j,y} + k (N_{i,x} N_{j,x} + N_{i,y} N_{j,y}) dxdy \right] \phi_j = 0 \quad (5)$$

$\Rightarrow$ over each element (e)

$$(D^e_{ij} + C^e_{ij}) \phi_j = 0$$

where $$D_{ij} = \int_\Omega k (N_{i,x} N_{j,x} + N_{i,y} N_{j,y}) dx\,dy \quad (7)$$

is diffusion matrix

and $$C_{ij} = \int_\Omega N_i (u N_{j,x} + v N_{j,y}) dx\,dy \quad (8)$$

is convection matrix

The volume integrals in (7), (8) are evaluated by Gaussian integration formula

each of these particular point represents some part of the element

$$\int_\Omega g(x,y)\,dx\,dy \simeq \sum_{m=1}^{M} g(x(\xi_m, \eta_m), y(\xi_m, \eta_m)) |J| w_m \quad (9)$$

where w_m is the weight associated with the integration points (ξ_m, η_m).

$\Rightarrow$ The numerical integration shown in (9) results in lumping at integration points. ξ_m, η_m are dragged along by the fluid motion described by (u, v) when evaluating the convection integral.

$\Rightarrow$ (1) Determination of the streamlines

(2) Establish the rule for node movement along the streamlines.

(1) Determinating the streamline

Velocity components u, v in physical plane (x, y) can be transformed to computational plane through one-to-one correspondence. (u_{ξ}, v_{η})

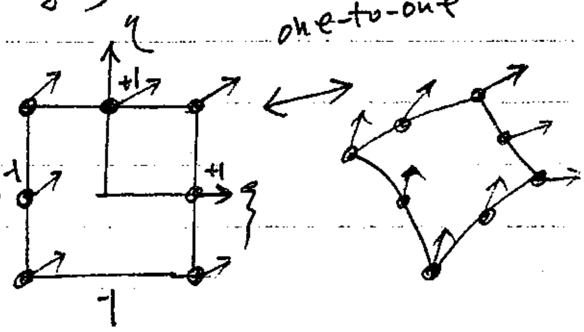
i.e.

$$u_{\xi} = l_{11} u + l_{12} v \qquad (P.76) \text{ in vector analysis}$$
$$v_{\eta} = l_{21} u + l_{22} v$$

where l_{ij}, $(i, j = 1, 2)$ represents direction cosines of (ξ, η) w.r.t. (x, y)

and

$$l_{11} = \underline{i} \cdot \underline{i}' = \xi_x = y_\eta / J$$
$$l_{12} = \underline{j} \cdot \underline{i}' = \xi_y = -x_\eta / J$$
$$l_{21} = \underline{i} \cdot \underline{j}' = \eta_x = -y_\xi / J$$
$$l_{22} = \underline{j} \cdot \underline{j}' = \eta_y = x_\xi / J$$



$$\Rightarrow \quad u_{\xi} = \frac{1}{J}(y_\eta u - x_\eta v) \qquad \text{(11)}$$
$$v_{\eta} = \frac{1}{J}(-y_\xi u + x_\xi v)$$

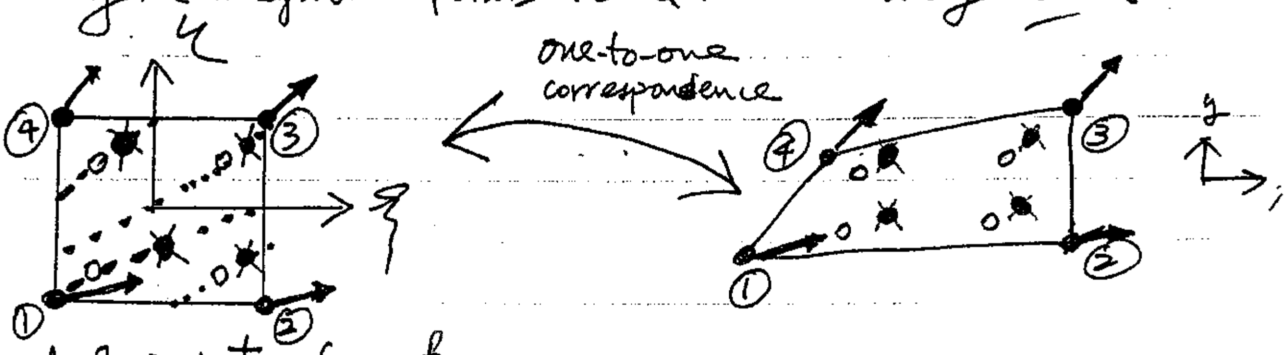
Isoparametric element assumption is made

$$\rightarrow \quad u_{\xi}(\xi, \eta) = \sum_{i=1}^{4} N_i(\xi, \eta)\, u_{\xi, i}$$
$$v_{\eta}(\xi, \eta) = \sum_{i=1}^{4} N_i(\xi, \eta)\, v_{\eta, i}$$

where

$$u_{\xi, i} = u_{\xi}(\xi_i, \eta_i)$$
$$v_{\eta, i} = v_{\eta}(\xi_i, \eta_i)$$

at mesh nodes are calculated through equation (11) evaluated at (ξ_i, η_i) point upwind finite element scheme removes spurious oscillations that occurs in the solution of diffusion convection equations especially for large Peclet no. flows. $\Longrightarrow$ It is proposed to move the integration nodes along the streamlines before evaluating the convection matrices $\underline{C}^e$. ie, moving the integration points to ✱ in evaluating $\underline{C}^e$.



- ● nodal points for unknowns
- ○ Integration points
- ✱ upwind integration points
- ····· streamline

The streamline $(\xi = f(t), \eta = g(t))$ where t varies from 0 to t', goes through integration point (ξ_m, η_m) and follows downward is governed by the following equations

(Maybe solved by second-order Runge-Kutta method)

$$\begin{cases} \dfrac{df}{dt} = \displaystyle\sum_{i=1}^{4} N_i(f(t), g(t))\, u_{\xi, i} \\[2ex] \dfrac{dg}{dt} = \displaystyle\sum_{i=1}^{4} N_i(f(t), g(t))\, v_{\eta, i} \end{cases}$$

This differential system produces the streamline below the point (ξ_m, η_m). The upper part of the streamline can be obtained by inverting the signs of $(u_{\xi, i}, v_{\eta, i})$

with $f(t=0) = \xi_m$ and $g(t=0) = \eta_m$

and $|f(t')| = 1$; $|g(t')| = 1$ *parallel computation

(The streamline crosses the border of the element)

(2) Establish the rule for node movement along the streamlines

One-dimensional example (for simplicity)

$$\begin{cases} -k\phi_{xx} + v\phi_x = 0 & 0<x<1 \\ \phi(0)=\phi_0, \quad \phi(1)=\phi_1 \end{cases} \quad \text{---(12)}$$

The coordinate transformation $x \to s$ is made (where $-1 \le s \le 1$)

by $x = x_i\left(\frac{1-s}{2}\right) + x_{i+1}\left(\frac{1+s}{2}\right)$ $\qquad x_1 = 0 < x_2 < \cdots < x_{n+1} = 1$

$x_{i+1} - x_i = h_i$

→ The evaluation of (7), (8) in one-dim case will be exact by one-point Gauss integration point. $R_i = \frac{1}{2}(x_i + x_{i+1})$

The evaluation of convection matrix C_{ij} is performed by introducing upwind coefficient α_i ($0 \le \alpha_i \le 1$) such that the integration point is moved to R_i' by

$$R_i' - R_i = \alpha_i \frac{h_i}{2}$$

Equation (5) becomes (T.J.R. Hughes, 1978) (in one-dim case)

$$\left[-\frac{v}{2}(R_{i-1})(1+\alpha_{i-1}) - \frac{k}{h_{i-1}}\right]\phi_{i-1} + \left[v(R_{i-1})\frac{1+\alpha_{i-1}}{2} - v(R_i)\frac{1-\alpha_i}{2} + \frac{k}{h_{i-1}} + \frac{k}{h_i}\right]\phi_i$$

$$+\left[v(R_i)\frac{1-\alpha_i}{2} - \frac{k}{h_i}\right]\phi_{i+1} = 0 \quad \text{---(13)}$$

<Note>: when v, k, h_i, α_i are constant, this relation reduces to that by Christie et. al.

Since 1-D case for (12) and $\phi_0 : 1$, $\phi_1 : 0$

$$\sum_i \int_{x_i}^{x_{i+1}} \left(k\phi_x N_{i,x} + v\phi_x N_i\right) dx = 0 \quad \text{---(14)}$$

if $\phi(x) = \sum_i \phi_i N_i(x)$

$$\rightarrow \int_{x_i}^{x_{i+1}} \left(k\,\phi_x N_{i,x} + v\,\phi_x N_i \right) dx$$

$$= h_i \left\{ k\,\phi_x N_{i,x} + v\,\phi_x N_i \right\}\Big|_P$$

$$- \alpha(P)\,\frac{h_i^2}{2} \left\{ k\,\phi_{xx} N_{i,x} + k\,\phi_x N_{i,xx} + v\,\phi_{xx} N_i + v\,\phi_x N_{i,x} \right\}\Big|_{x_{i+1}}$$

Since $\displaystyle\int_{x_i}^{x_{i+1}} f(x)\,dx = h\,f(P) - \alpha(P)\,\frac{h^2}{2}\,f'(x_{i+1/2})$

where $f(x)$ is a linear function in $[x_i, x_{i+1}]$

$h = x_{i+1} - x_i$; $x_{i+1/2} = \frac{1}{2}(x_i + x_{i+1})$

$\alpha(P) = \frac{2}{h}(P - x_{i+1/2})$

P is a point in $[x_i, x_{i+1}]$

$\because f(P) = f(x_{i+1/2}) + \alpha(P)\,\frac{h}{2}\,f'(\xi)$ where $x_{i+1/2} \le \xi \le P$, since f is linear $\Rightarrow f'(\xi) = f'(x_{i+1/2})$ and $\int_{x_i}^{x_{i+1}} f(x)\,dx = f(\cdot)\,h$

But $\displaystyle h_i \left\{ k\,\phi_x N_{i,x} + v\,\phi_x N_i \right\}\Big|_P = \int_{x_i}^{x_{i+1}} \left\{ \left(k + \frac{\alpha(P)\,v\,h_i}{2}\right) \phi_x N_{i,x} + v\,\phi_x N_i \right\} dx$

Since $\displaystyle\int_{x_i}^{x_{i+1}} \frac{d\phi}{dx}\frac{dN_i}{dx}\,dx = h_i\,\frac{d\phi}{dx}\,\frac{dN_i}{dx}\Big|_{x_{i+1}}$ (since $\phi(x)$ and $N_i(x)$ are piecewise linear)

$\rightarrow$ Introducing numerical diffusion term

$$\int_{x_i}^{x_{i+1}} \frac{\alpha(P)\,v\,h_i}{2}\,\frac{d\phi}{dx}\,\frac{dN_i}{dx}\,dx$$

For uniform mesh, the choice of

$\alpha = \cot\left(\frac{Pe}{2}\right) - \frac{2}{Pe}$ leads to exact value of ϕ at nodes.

- Can be generalized without difficulty to multiple dimensions
- Reliable for whatever grid, Pe, vel. field
- Easily implemented if numerical integration is employed

False diffusion (Cross-wind diffusion)

Most upwind and hybrid scheme suffer from severe false diffusion when the flow direction is at an angle to the grid lines.

False diffusion tends to augment the transport of dependent variable ϕ in the direction normal to the local streamline.

False diffusion results usually from the treatment of multi-dimensional convection-diffusion problems by means of locally one-dimensional discretization formulas along the grid lines. The amount of false diffusion increases as the inclination of the streamlines w.r.t. the grid lines increases.

Note: QUICK scheme developed by Leonard uses quadratic upstream interpolation procedure. It utilizes high-order Taylor series expansions to obtain a f.d. scheme which has very little false diffusion.

Numerical Heat transfer, 1985 S. Ramadhyani, S.V. Patankar

Shape function for convection-diffusion problems.

Consider one-dimensional convection-diffusion equation:

$$\frac{d}{dx}\left(\rho u \phi - \Gamma \frac{d\phi}{dx} \right) = 0$$

where $x = 0 \quad \phi = \phi_0$ —— (1)

$x = L \quad \phi = \phi_1$

The analytic solution for (1) is

$$\frac{\phi - \phi_0}{\phi_1 - \phi_0} = \frac{\exp(Px/L) - 1}{\exp(P) - 1} \qquad \text{—— (2)}$$

where $P \equiv \rho u L / \Gamma$ is Peclet number.

The expotential behavior in (2) indicates that the shape function for (1) in finite element approximation is

$$N_i = a_i + b_i \left[\exp\left(P_\delta x / \delta \right) - 1 \right]$$

where $P_\delta \equiv \rho u \delta / \Gamma$ is determined based on the length of the element δ.

a_i, b_i are determined by

$$N_i = \begin{cases} 0 & \text{if } x = x_{i-1} \\ 1 & \phantom{\text{if }} x = x_i \end{cases} \quad \text{for element } (i-1, i)$$

$$N_i = \begin{cases} 1 & \text{if } x = x_i \\ 0 & \phantom{\text{if }} x = x_{i+1} \end{cases} \quad \text{for element } (i, i+1)$$

Note:
- In the limit of very low P_δ, the expotential shape function reduces to standard piecewise-linear shape function.
- In the limit of very high P_δ, the shape function becomes unbalanced near the downstream side.

A two-dimensional shape function for rectangular element is developed analogously with requirements:

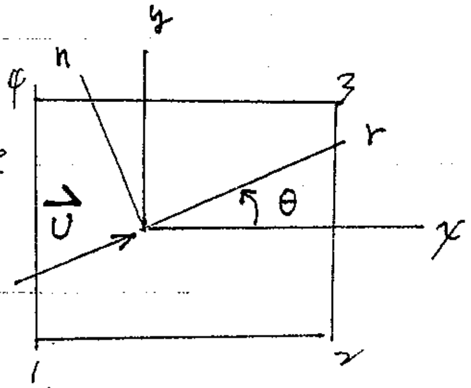
(a) As $P_\delta \to 0$, the shape function reduces to standard bilinear shape function.

(b) 2-d. shape function has an exponential profile in the streamline direction to realistically depict the variation of dependent variable.

(c) Capable of representing a purely 1-d situation when the flow is oriented along either of the coordinate direction.

(b) $\Longrightarrow$ A local coordinate system is then constructed with r as flow direction and n the coordinate normal to r.

The r-n coordinate system is related to the local x-y coordinate system



$$\begin{pmatrix} r \\ n \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Longrightarrow N_i = \overbrace{a_i + b_i\left[\exp\left(\frac{\rho U r}{\Gamma}\right) - 1\right]}^{\text{flow-directed shape function}} + c_i n$$

$$+ \underbrace{d_i\left[\exp\left(\frac{P_x\, x}{L_1}\right) - 1\right]\left[\exp\left(\frac{P_y\, y}{L_2}\right) - 1\right]}_{\text{for condition (a)}} \qquad (3)$$

where $P_x \equiv \rho u L_1 / \Gamma$

$P_y = \rho v L_2 / \Gamma$

u, v are components of $\vec{U}$ in x, y directions respectiv

L_1, L_2 are the corresponding lengths of the sides of the element.

Note : • As Peclet no. is considered, N_i reduces to standard bilinear form

$$N_i = \alpha_i + \beta_i x + \gamma_i y + \delta_i xy$$

since $\lim_{\theta \to 0} \exp(\theta) - 1 = 0$

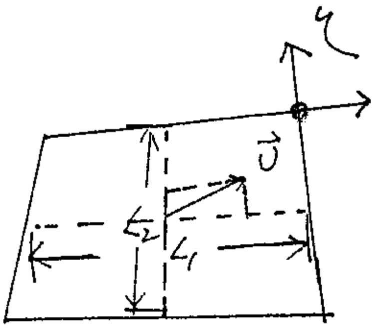
• As Peclet no is high, the expotential terms are large positive → leading to computational difficulties ⟹ The origins of both r-n and x-y coordinates are assumed to be fixed at the most downstream corner of the rectangular element which can ensure the arguments of expotential terms are always negative.

If elements are quadrilaterals of arbitrary shape,

$$N_i = a_i + b_i\left[\exp\left(\rho U r/\Gamma \;\; -1\right)\right] + c_i n$$

$$+ d_i\left[\exp(P_\xi \xi) - 1\right]\left[\exp(P_\eta\, \eta - 1)\right]$$

where P_ξ, P_η are Peclet no in ξ, η directions based on components of $\underline{U}$ in ξ, η directions.



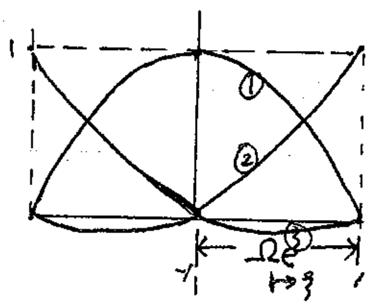
The origin of ξ-η coordinate is assumed to be at the most downstream corner of the element.

Upwind finite element method (Upwind QUICK PG) for convection-dominated problems

1989 P.M. Steffler Canada

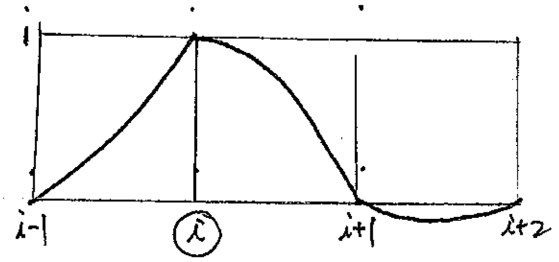
Physically, the value of a convection-dominated variable within an element depends more heavily on upstream nodal values than on the downstream nodal values.

Upwind basis functions $f(\xi)$: (Assume flow direction is positive)

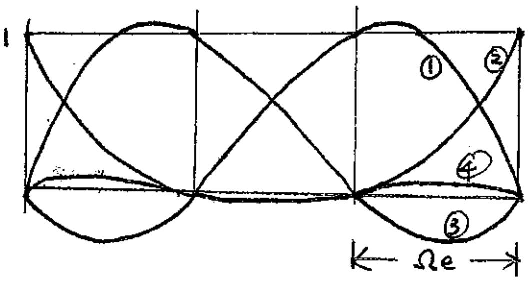


$$fe^{①}(\xi) = (3+\xi)(1-\xi)/4$$
$$fe^{②}(\xi) = (1+\xi)(3+\xi)/8 \quad , \quad -1 \le \xi \le 1$$
$$fe^{③}(\xi) = -(1-\xi)(1+\xi)/8$$

local quadratic

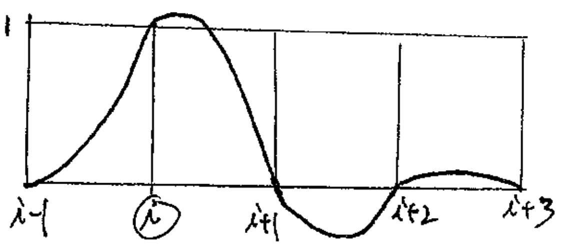


global quadratic



$$fe^{①}(\xi) = (5+\xi)(3+\xi)(1-\xi)/16$$
$$fe^{②}(\xi) = (1+\xi)(3+\xi)(5+\xi)/48 \quad , \quad -1 \le \xi \le 1$$
$$fe^{③}(\xi) = -(1-\xi)(1+\xi)(5+\xi)/16$$
$$fe^{④}(\xi) = -(1-\xi)(1+\xi)(3+\xi)/48$$

local cubic



global cubic

Initial boundary value problem

(A) Scalar hyperbolic conservation law:

Consider hyperbolic conservation law

$$u_t + \sum_{i=1}^{d} (f_i(u))_{x_i} = 0$$

$u = (u_1, \cdots, u_m)^T$; $x = (x_1, \cdots, x_d)$ where $d = m = 1$.

I.C.: $u(x,0) = u^0(x)$, $x \in \mathbb{R}$

B.C: $u(a,t) = g(t)$, $x_{-1/2} = a < x < b$ where $b = \infty$

Consider 1-D, steady, convection-diffusion b.v.p:

$$\begin{cases} c\,\partial\phi/\partial x - \partial/\partial x\left(\nu\,\partial\phi/\partial x\right) = 0 & x \in (0, L) \qquad (1) \\ \phi(x=0) = 0 \\ \phi(x=L) = 1 \end{cases}$$

Assume $\phi_h = f\,\Phi$ is the approximation solution to (1).

The weak statement of (1) becomes

$$\int (c\, f_x\, \Phi - \nu\, f_{xx}\, \Phi)\, v\, dx = 0$$

where v is test function in weighted residual procedures.

Integrating by parts

$$\int c\,\Phi(f v)_x - c\,\Phi f v_x - \nu\,\Phi(f_x v)_x + \nu\,\Phi f_x v_x \; dx = 0$$

$$\left[c\,\Phi(f v)\Big|_0^L - \left(\int_0^L c\, f\, v_x\, dx\right)\Phi\right] - \nu\,\Phi\,\cancel{(f_x v)\Big|_0^L} + \left(\int_0^L \nu\, f_x\, v_x\, dx\right)\Phi = 0$$

$$\left[\ \ldots\ \right] = \left(\int_0^L c\, v\, f_x\, dx\right)\Phi$$

$$\rightarrow \quad (C + K)\,\Phi = 0$$

where $C = \int_0^L c\, v\, \frac{df}{dx}\, dx$

$$K = \int_0^L \nu\, \frac{dv}{dx}\, \frac{df}{dx}\, dx$$

Over each element e, define

$$C_e = \int_{-1}^{1} v_e\, \frac{df_e}{d\xi}\, d\xi$$

$$K_e = \int_{-1}^{1} \frac{dv_e}{d\xi}\, \frac{df_e}{d\xi}\, d\xi$$

(a) Quadratic upwind Bubnov-Galerkin method (QUBG)

$$C_e = \begin{bmatrix} 0 & \frac{1}{12} & \frac{-1}{12} \\ -\frac{1}{12} & -\frac{1}{2} & \frac{7}{12} \\ \frac{1}{12} & -\frac{7}{12} & \frac{1}{2} \end{bmatrix}_{3\times 3}, \quad K_e = \begin{bmatrix} \frac{1}{12} & -\frac{1}{6} & \frac{1}{12} \\ -\frac{1}{6} & \frac{4}{3} & -\frac{7}{6} \\ \frac{1}{12} & -\frac{7}{6} & \frac{13}{12} \end{bmatrix}_{3\times 3}$$

$$i-2 \quad i-1 \quad i \quad i+1 \quad i+2$$

For convective part:

$$\frac{c}{12}\begin{bmatrix} 0 & 1 & -1 \\ -1 & -6 & 7 \\ -1 & -7 & 6 \end{bmatrix} \begin{bmatrix} \Phi_i \\ \Phi_{i+1} \\ \Phi_{i+2} \end{bmatrix}, \begin{bmatrix} \Phi_{i-1} \\ \Phi_i \\ \Phi_{i+1} \end{bmatrix}, \begin{bmatrix} \Phi_{i-2} \\ \Phi_{i-1} \\ \Phi_i \end{bmatrix}$$

$$\rightarrow \begin{array}{lccccc} \Phi_{i-2} & & & 1 & & 1 \\ \Phi_{i-1} & & -1 & -7 & & -8 \\ \Phi_i & 0 & -6 & 6 & \rightarrow & 0 \\ \Phi_{i+1} & 1 & 7 & & & 8 \\ \Phi_{i+2} & -1 & & & & -1 \end{array} \rightarrow \frac{c}{12}\left(\Phi_{j-2} - 8\Phi_{j-1} + 8\Phi_{j+1} - \Phi_{i+2}\right)$$

For diffusive part:

$$\frac{\nu}{12}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 16 & -14 \\ 1 & -14 & 13 \end{bmatrix} \begin{bmatrix} \Phi_i \\ \Phi_{i+1} \\ \Phi_{i+2} \end{bmatrix}, \begin{bmatrix} \Phi_{i-1} \\ \Phi_i \\ \Phi_{i+1} \end{bmatrix}, \begin{bmatrix} \Phi_{i-2} \\ \Phi_{i-1} \\ \Phi_i \end{bmatrix}$$

$$\rightarrow \begin{array}{lccccc} \Phi_{i-2} & & & 1 & & 1 \\ \Phi_{i-1} & & -2 & -14 & & -16 \\ \Phi_i & 1 & 16 & 13 & \rightarrow & 30 \\ \Phi_{i+1} & -2 & -14 & & & -16 \\ \Phi_{i+2} & 1 & & & & 1 \end{array} \rightarrow \frac{\nu}{12}\left(\Phi_{j-2} - 16\Phi_{j-1} + 30\Phi_j - 16\Phi_{j+1} + \Phi_{j+2}\right)$$

$\Longrightarrow$ The discretized equation

$$\frac{c}{12}\left(\Phi_{j-2} - 8\Phi_{j-1} + 8\Phi_{j+1} - \Phi_{j+2}\right) + \frac{\nu}{12h}\left(\Phi_{j-2} - 16\Phi_{j-1} + 30\Phi_j - 16\Phi_{j+1} + \Phi_{j+2}\right) = 0$$

recognized as 4th order central difference for the first and second derivatives

(b) Quadratic upwind Petrov-Galerkin method. (QUPG)
Using quadratic upwind basis function and standard linear test function

it results in $C_e = \frac{c}{12}\begin{bmatrix} -1 & -4 & 5 \\ 1 & -8 & 7 \end{bmatrix}$

$$K_e = \frac{\nu}{12}\begin{bmatrix} 0 & 12 & -12 \\ 0 & -12 & 12 \end{bmatrix}$$

For convective part:

$$\frac{c}{12}\begin{bmatrix} -1 & -4 & 5 \\ 1 & -8 & 7 \end{bmatrix} \begin{bmatrix} \Phi_i \\ \Phi_{i+1} \\ \Phi_{i+2} \end{bmatrix}, \begin{bmatrix} \Phi_{i-1} \\ \Phi_i \\ \Phi_{i+1} \end{bmatrix}, \begin{bmatrix} \Phi_{i-2} \\ \Phi_{i-1} \\ \Phi_i \end{bmatrix}$$

$$\begin{array}{llll} \Phi_{i-2} & & 1 & \\ \Phi_{i-1} & -1 & -8 & \\ \Phi_i & -4 & 7 & \\ \Phi_{i+1} & 5 & & \\ \Phi_{i+2} & & & \end{array} \rightarrow \begin{array}{c} 1 \\ -9 \\ 3 \\ 5 \\ \end{array} \rightarrow \frac{c}{12}\left(\Phi_{j-2} - 9\Phi_{j-1} + 3\Phi_j + 5\Phi_{j+1}\right)$$

For diffusion part:

$$\frac{\nu}{12}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 12 & -12 \\ 0 & -12 & 12 \end{bmatrix} \begin{bmatrix} \Phi_i \\ \Phi_{i+1} \\ \Phi_{i+2} \end{bmatrix}, \begin{bmatrix} \Phi_{i-1} \\ \Phi_i \\ \Phi_{i+1} \end{bmatrix}, \begin{bmatrix} \Phi_{i-2} \\ \Phi_{i-1} \\ \Phi_i \end{bmatrix}$$

$$\begin{array}{llll} \Phi_{i-2} & & 0 & \\ \Phi_{i-1} & 0 & -12 & \\ \Phi_i & 12 & 12 & \\ \Phi_{i+1} & -12 & & \\ \Phi_{i+2} & & & \end{array} \rightarrow \begin{array}{c} 0 \\ -12 \\ 24 \\ -12 \\ \end{array} \rightarrow \nu\left(-\Phi_{j-1} + 2\Phi_j - \Phi_{j+1}\right)$$

$\Longrightarrow$ The discretized equation:

$$\frac{c}{12}\left(\Phi_{j-2} - 9\Phi_{j-1} + 3\Phi_j + 5\Phi_{j+1}\right) - \frac{\nu}{h}\left(-\Phi_{j-1} + 2\Phi_j - \Phi_{j+1}\right) = 0$$

(n) Cubic upwind Petrov-Galerkin method (CUPG)

$$C_e = \frac{c}{24}\begin{bmatrix} 1 & -5 & -5 & 9 \\ -1 & 5 & -19 & 15 \end{bmatrix}, \quad K_e = \nu\begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix},$$

for convective part:

$$\frac{c}{24}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -5 & 7 & -3 \\ -1 & 5 & -7 & 3 \end{bmatrix} \begin{bmatrix} \Phi_{j-3} \\ \Phi_{j-2} \\ \Phi_{j-1} \\ \Phi_{j} \end{bmatrix}, \begin{bmatrix} \Phi_{j-2} \\ \Phi_{j-1} \\ \Phi_{j} \\ \Phi_{j+1} \end{bmatrix}, \begin{bmatrix} \Phi_{j-1} \\ \Phi_{j} \\ \Phi_{j+1} \\ \Phi_{j+2} \end{bmatrix}, \begin{bmatrix} \Phi_{j} \\ \Phi_{j+1} \\ \Phi_{j+2} \\ \Phi_{j+3} \end{bmatrix}$$

Φ_{j-3}	-1			-1
Φ_{j-2}	5	1		6
Φ_{j-1}	-19	-5		-24
Φ_{j}	15	-5	$\rightarrow$	10
Φ_{j+1}		9		9
Φ_{j+2}				
Φ_{j+3}				

$$\rightarrow \frac{c}{24}\left(-\Phi_{j-3} + 6\Phi_{j-2} - 24\Phi_{j-1} + 10\Phi_{j} + 9\Phi_{j+1}\right)$$

For diffusive part

$$\nu \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \Phi_{j-3} \\ \Phi_{j-2} \\ \Phi_{j-1} \\ \Phi_{j} \end{bmatrix}, \begin{bmatrix} \Phi_{j-2} \\ \Phi_{j-1} \\ \Phi_{j} \\ \Phi_{j+1} \end{bmatrix}, \begin{bmatrix} \Phi_{j-1} \\ \Phi_{j} \\ \Phi_{j+1} \\ \Phi_{j+2} \end{bmatrix}, \begin{bmatrix} \Phi_{j} \\ \Phi_{j+1} \\ \Phi_{j+2} \\ \Phi_{j+3} \end{bmatrix}$$

Φ_{j-3}	0			0
Φ_{j-2}	0	0		0
Φ_{j-1}	-1	0		-1
Φ_{j}	1	1	$\rightarrow$	2
Φ_{j+1}		-1		-1
Φ_{j+2}				
Φ_{j+3}				

$$\rightarrow \frac{\nu}{h}\left(-\Phi_{j-1} + 2\Phi_{j} - \Phi_{j+1}\right)$$

$\Longrightarrow$ The discretized equation

$$\frac{c}{24}\left(-\Phi_{j-3} + 6\Phi_{j-2} - 24\Phi_{j-1} + 10\Phi_{j} + 9\Phi_{j+1}\right)$$

$$+\frac{\nu}{h}\left(-\Phi_{j-1} + 2\Phi_{j} - \Phi_{j+1}\right) = 0$$

The diffusion terms are unaffected compared with standard linear finite element formulation.

The corresponding modified equation is (4th-order accurate)

$$c\,\partial\phi_h/\partial x - \nu\,\partial^2\phi_h/\partial x^2 + h^2/12\;\partial^2/\partial x^2\left(c\,\partial\phi_h/\partial x - \nu\,\partial^2\phi_h/\partial x^2\right)$$
$$+\,h^4/720\;(21\,c\,\partial^5\phi_h/\partial x^5 - 2\nu\,\partial^6\phi_h/\partial x^6) + O(h^5) = 0$$

__Note__:

The modified equation of QUPG formulation is

$$c\,\partial\phi_h/\partial x - \nu\,\partial^2\phi_h/\partial x^2 + h^2/12\;\partial^2/\partial x^2\left(c\,\partial\phi_h/\partial x - \nu\,\partial^2\phi_h/\partial x^2\right)$$
$$+\,h^3/12\;c\,\partial^4\phi_h/\partial x^4 + h^4/720\;(9c\,\partial^5\phi_h/\partial x^5 - 2\nu\,\partial^6\phi_h/\partial x^6) + O(h^5) = 0$$

Linear advection equation

Consider $\partial\phi/\partial t + c\,\partial\phi/\partial x = 0$

$$\phi(x,0) = \phi_0(x)$$

and periodic b.c.

The standard weak statement results in

$$M\,\frac{d\Phi}{dt} + C\,\Phi = 0$$

Where $M = \int_0^L v f\,dx$

$$C = \int_0^L c\,v\,df/dx\;dx$$

For __QUPG__ $\quad M_e = \frac{1}{24}\begin{bmatrix} -1 & 10 & 3 \\ -1 & 6 & 7 \end{bmatrix}$

The resulting modified equation is

$$\frac{\partial\phi_h}{\partial t} + c\frac{\partial\phi_h}{\partial x} + \frac{h^2}{12}\frac{\partial^2}{\partial x^2}\left(\frac{\partial\phi_h}{\partial t} + c\frac{\partial\phi_h}{\partial x}\right) + \frac{h^3}{24}\frac{\partial^3}{\partial x^3}\left(\frac{\partial\phi_h}{\partial t} + c\frac{\partial\phi_h}{\partial x}\right)$$
$$-\frac{h^4}{72}\frac{\partial^4}{\partial x^4}\left(\frac{\partial\phi_h}{\partial t} + c\frac{\partial\phi_h}{\partial x}\right) + \frac{h^4}{720}\,c\,\frac{\partial^5\phi_h}{\partial x^5} + O(h^5) = 0$$

For <u>COPG</u>, $M_e = \frac{1}{6}\begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} + \frac{1}{24}\begin{bmatrix} 0 & -1 & 2 & -1 \\ 0 & -1 & 2 & -1 \end{bmatrix} + \frac{1}{360}\begin{bmatrix} 7 & -21 & 21 & -7 \\ 8 & -24 & 24 & -8 \end{bmatrix}$

The resulting modified equation is

$$\frac{\partial \phi_h}{\partial t} + c\frac{\partial \phi_h}{\partial x} + \frac{h^2}{12}\frac{\partial^2}{\partial x^2}\left(\frac{\partial \phi_h}{\partial t} + c\frac{\partial \phi_h}{\partial x}\right) + \frac{7h^4}{240}\frac{\partial^4}{\partial x^4}\left(\frac{\partial \phi_h}{\partial t} + c\frac{\partial \phi_h}{\partial x}\right)$$
$$- \frac{h^5}{45}\frac{\partial^5}{\partial x^5}\left(\frac{\partial \phi_h}{\partial t} + c\frac{\partial \phi_h}{\partial x}\right) + \frac{h^5}{720} c \frac{\partial^6 \phi_h}{\partial x^6} + O(h^6) = 0$$

* Conclusion – This formulation does not introduce new internal d.o.f for obtaining higher-order accuracy, it is achieved by including functions based on nodal values exterior and upwind to the element domain.

and QU
SUPG schemes
(Streamline Upwind Petrov-Galerkin method)

One-dimensional upwind finite element scheme

Consider $u\phi' = (k\phi')' + f$ in $[0, L]$

$\phi(0) = g$ on Γ_g

$k(L)\phi'(L) = h$ on Γ_h $\quad \left(\begin{matrix} \Gamma_g \cap \Gamma_h = \{0\} \\ \Gamma_g \cup \Gamma_h = \Gamma \end{matrix} \right)$ (1)

A weak formulation for (1) leads to

$$\int_0^L (w u \phi' + w' k \phi')\, dx = \int_0^L w f \, dx + w(L) h \quad \text{—(2)}$$

Where $w(0) = 0$

Assume u and k are constant and $f = 0$, one can develop the discretized equation (2) at interior points using piecewise linear shape functions and unsymmetric quadrature rule

$$\frac{u}{2h}(\phi_{i+1} - \phi_{i-1}) = (k + \tilde{k}) \frac{1}{h^2}(\phi_{i+1} - 2\phi_i + \phi_{i-1}) \quad \text{—(3)}$$

Where h is element length

$\tilde{k} = \frac{uh}{2}\tilde{\xi}$ is artificial viscosity

$\tilde{\xi} = (\coth\alpha) - \frac{1}{\alpha}$ is quadrature point —(3)'

$\alpha = \frac{uh}{2k}$ is element Peclet number

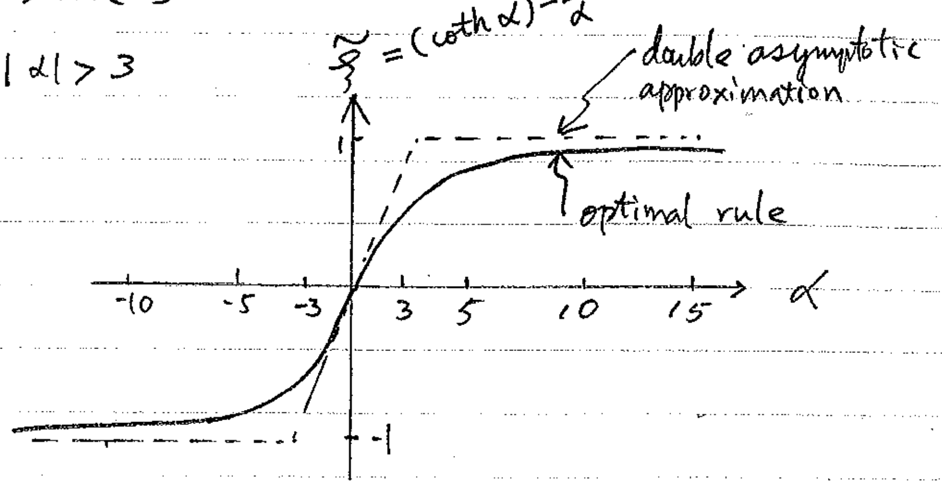
Note – Another approach involves adding the artificial to the weak form

$$\int_0^L [w u \phi' + w'(k + \tilde{k})\phi']\, dx = \int_0^L w f \, dx + w(L) h \quad \text{—(3)''}$$

and subsequently applying the standard Galerkin / finite element discretization.

– to minimize the cost of large calculations, one can employ "double asymptotic approximation of (3)' by

$$\tilde{\xi} = \begin{cases} \alpha/3 & ; \ -3 \le \alpha \le 3 \\ \operatorname{sgn} \alpha & ; \ |\alpha| > 3 \end{cases}$$



$u\phi' = (u\phi)'$
$= \sigma^a_{\lambda,\lambda}$
$\sigma^a_\lambda = u\phi$

— expression (3)'' can be rewritten as

$$\int_0^L \left[(w u \phi' + w' \tilde{k} \phi') + w' k \phi' \right] dx = \int_0^L w f \, dx + w(L) h$$

$$\int_0^L u\phi' \left(w + \frac{w' \tilde{k} u}{u^2} \right) + w' k \phi' \, dx = \int_0^L w f \, dx + w(L) h$$

$$\int_0^L u\phi' (w + \tau \, u w') + w' k \phi' \, dx = \int_0^L w f \, dx + w(L) h$$

$$\rightarrow \int_0^L u\phi' (w + \sigma^a_{i,i}(w) \tau) + w' k \phi' \, dx = \int_0^L w f \, dx + w(L) h$$

$$\rightarrow \int_0^L u\phi' \tilde{w} + w' k \phi' \, dx = \int_0^L w f \, dx + w(L) h \quad \text{—— (4)}$$

where $\tilde{w} = w + p \implies$ Petrov-Galerkin finite element.

$p = \tau \, \sigma^a_{i,i}(w)$ —— (5)

$\tau = \tilde{k}/u^2$

$\sigma^a_{i,i}(w) = u w'$

the degree of upwinding depends on element Peclet number α and the choice of quadrature point.

Multi-dimensional upwind finite element schemes

<A> Quadrature Upwind (QU) scheme

— generalize the 1-D upwind scheme to basic isoparametric multi-dimensional elements by employing a one-point quadrature rule on the advection term

For bi-linear quadrilateral element, the location of the quadrature point is defined by

$$\underline{\tilde{\xi}} = \begin{Bmatrix} \tilde{\xi} \\ \tilde{\eta} \end{Bmatrix}$$

where
$$\tilde{\xi} = (\coth \alpha_\xi) - \frac{1}{\alpha_\xi} \qquad \tilde{\eta} = (\coth \alpha_\eta) - \frac{1}{\alpha_\eta}$$

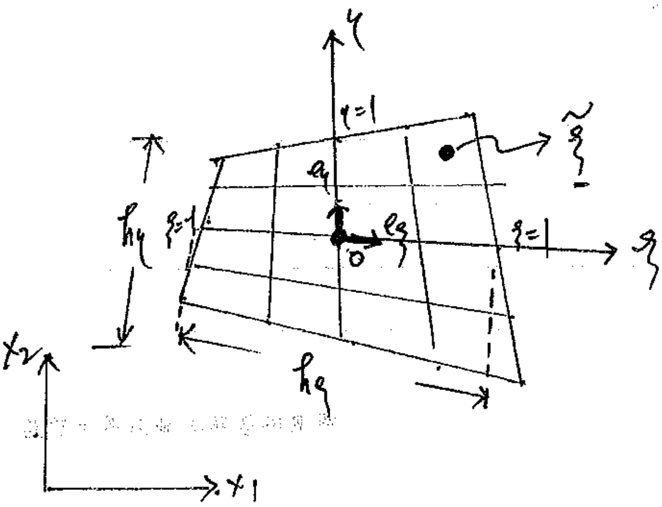
$$\alpha_\xi = \frac{u_\xi h_\xi}{2k} \qquad \alpha_\eta = \frac{u_\eta h_\eta}{2k}$$

$$u_\xi = \underline{e}_\xi \cdot \underline{u} \qquad u_\eta = \underline{e}_\eta \cdot \underline{u}$$

$\underline{e}_\xi$, $\underline{e}_\eta$ are unit vector

h_ξ, h_η are element lengths

u, k are evaluated at the origin of the element o.



QU results in excessive crosswind diffusion (or false diffusion)

<B> Streamline Upwind Petrov-Galerkin scheme (SUPG).

— For avoiding spurious crosswind diffusion
(false)

In the case of bilinear isoparametric quadrilateral

$\tilde{w} = w + p$

$p = \tau \, \sigma^{a}_{i,i}(w)$

$\tau = \tilde{k}/u_i u_i$

$\tilde{k} = (\tilde{\xi}\, u_\xi h_\xi + \tilde{\eta}\, u_\eta h_\eta)/2$

$\tilde{\xi} = (\coth \alpha_\xi) - 1/\alpha_\xi \quad ; \quad \tilde{\eta} = (\coth \alpha_\eta) - 1/\alpha_\eta$

$\alpha_\xi = u_\xi h_\xi / 2k \quad ; \quad \alpha_\eta = u_\eta h_\eta / 2k$

$u_\xi = \underline{e}_\xi \cdot \underline{u} \quad ; \quad u_\eta = \underline{e}_\eta \cdot \underline{u}$

Apply the analog of (3'') for multidimensional case

$\rightarrow \int_\Omega w u_i \phi_{,i} + w_{,i}(k_{ij} + \tilde{k}_{ij})\phi_{,j} \, d\Omega = \int_\Omega w f \, d\Omega + \int_{\Gamma_h} w h \, d\Gamma$

The artificial diffusion term is combined with the advection term

$\rightarrow \int_\Omega (w u_i \phi_{,i} + w_{,i} \tilde{k}_{ij} \phi_{,j}) + w_{,i} k_{ij} \phi_{,j} \, d\Omega = \int_\Omega w f \, d\Omega + \int_{\Gamma_h} w h \, d\Gamma$

$\int_\Omega u_i \phi_{,i} \left(w + \frac{w_{,j} \tilde{k}_{ji} u_i}{u_i u_i}\right) + w_{,i} k_{ij} \phi_{,j} \, d\Omega = \int_\Omega w f \, d\Omega + \int_{\Gamma_h} w h \, d\Gamma$

$\int_\Omega u_i \phi_{,i} \left(w + \frac{w_{,j} \tilde{k}_{ji} \|\underline{u}\| u_i}{\|\underline{u}\|^2}\right) + w_{,i} k_{ij} \phi_{,j} \, d\Omega = \int_\Omega w f \, d\Omega + \int_{\Gamma_h} w h \, d\Gamma$

$\int_\Omega u_i \phi_{,i} \tilde{w} + w_{,i} k_{ij} \phi_{,j} \, d\Omega = \int_\Omega w f \, d\Omega + \int_{\Gamma_h} w h \, d\Gamma$

where $\tilde{w} = w + \left(\tilde{k}_{ji} w_{,i} \phi_j / \|\underline{u}\|\right)$

$\hat{u}_i \equiv u_i / \|\underline{u}\|$

$\|\underline{u}\|^2 \equiv u_i u_i$