Instantons in AdS/CFT duality

Pei-Ming Ho
Department of Physics
National Taiwan University

Yang-Mills theory

gauge potential $A = dx^\mu A_\mu^a(x)\tau_a$

Lie algebra $[\tau_a, \tau_b] = f_{abc}\tau_c$

field strength $F = dA + AA$

$$F = \frac{1}{2} dx^\mu dx^\nu F_{\mu\nu}^a(x)\tau_a$$

$$F_{\mu\nu} = [\partial_\mu + A_\mu, \partial_\nu + A_\nu]$$

Yang-Mills action

$$S_{\text{YM}} = \frac{1}{g_{\text{YM}}^2} \int \mathbf{Tr}\,(F\,{}^*F)$$

Variation $\delta A \rightarrow$ equation of motion.

Hodge duality = electromagnetic duality

$$F \longrightarrow {}^*F$$

$${}^*F_{\mu\nu} \equiv \pm\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F_{\alpha\beta}$$

(anti-) Self-dual = (anti-) Instanton

$$F = \pm {}^* F$$

$$\Rightarrow S_{\mathsf{YM}} = \frac{8\pi^2}{g_{\mathsf{YM}}^2} Q$$

Instanton number = topological charge

$$Q = \frac{-1}{16\pi^2} \int \mathbf{Tr}\,(FF) \in \mathbb{Z}$$

$$\mathbf{Tr}\,(FF) = d\,\mathbf{Tr}\left(AF - \frac{2}{3}AAA\right)$$

$$\delta Q = 0$$

- **Self-duality $\Rightarrow$ equation of motion**
- **YM instantons have vanishing energy-momentum tensor.**
- **Conformal group $SO(5,1)$ acts on the moduli space.**

1-instanton solution

$$A = \left(\frac{r^2}{r^2 + L^2}\right) U^{-1} dU$$

$$r^2 = x_\mu x^\mu \qquad (\mu = 1, \cdots, 4)$$

$$A \to U^{-1} dU, \qquad r \to \infty$$

$$U = \frac{x^4 - i x^a \sigma_a}{r} \qquad (i = 1, 2, 3)$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

1-instanton solution

$$F_{\mu\nu} = 4Z^2(x; x_0, L)\sigma_{\mu\nu}$$

$$\sigma_{4a} = \frac{1}{2}\sigma_a, \qquad \sigma_{ab} = \frac{1}{4i}[\sigma_a, \sigma_b]$$

$$Z(x; x_0, L) \equiv \frac{L}{L^2 + |x - x_0|^2}$$

$$\textbf{Tr}\ F_{\mu\nu}{}^{(*)}F^{\mu\nu} \propto Z^4(x; x_0, L).$$

AdS/CFT duality:

Type IIB superstring in $AdS_5 \times S^5$
$\simeq SU(N)$ super Yang-Mills theory in 4D.

Maldacena [98]; Gubser, Klebanov, Polyakov [98]; Witten [98]

$$(g, R)_{\mathrm{IIB}} \leftrightarrow (g_{\mathrm{YM}}, N)_{\mathrm{YM}}:$$

$$4\pi g = g_{\mathrm{YM}}^2, \qquad R^2/l_s^2 = 4\pi g\sqrt{N}.$$

$AdS_5 \to$ Euclidean AdS_5

$EAdS_5$ **embedded in** $\mathbb{R}^{5,1}$**:**

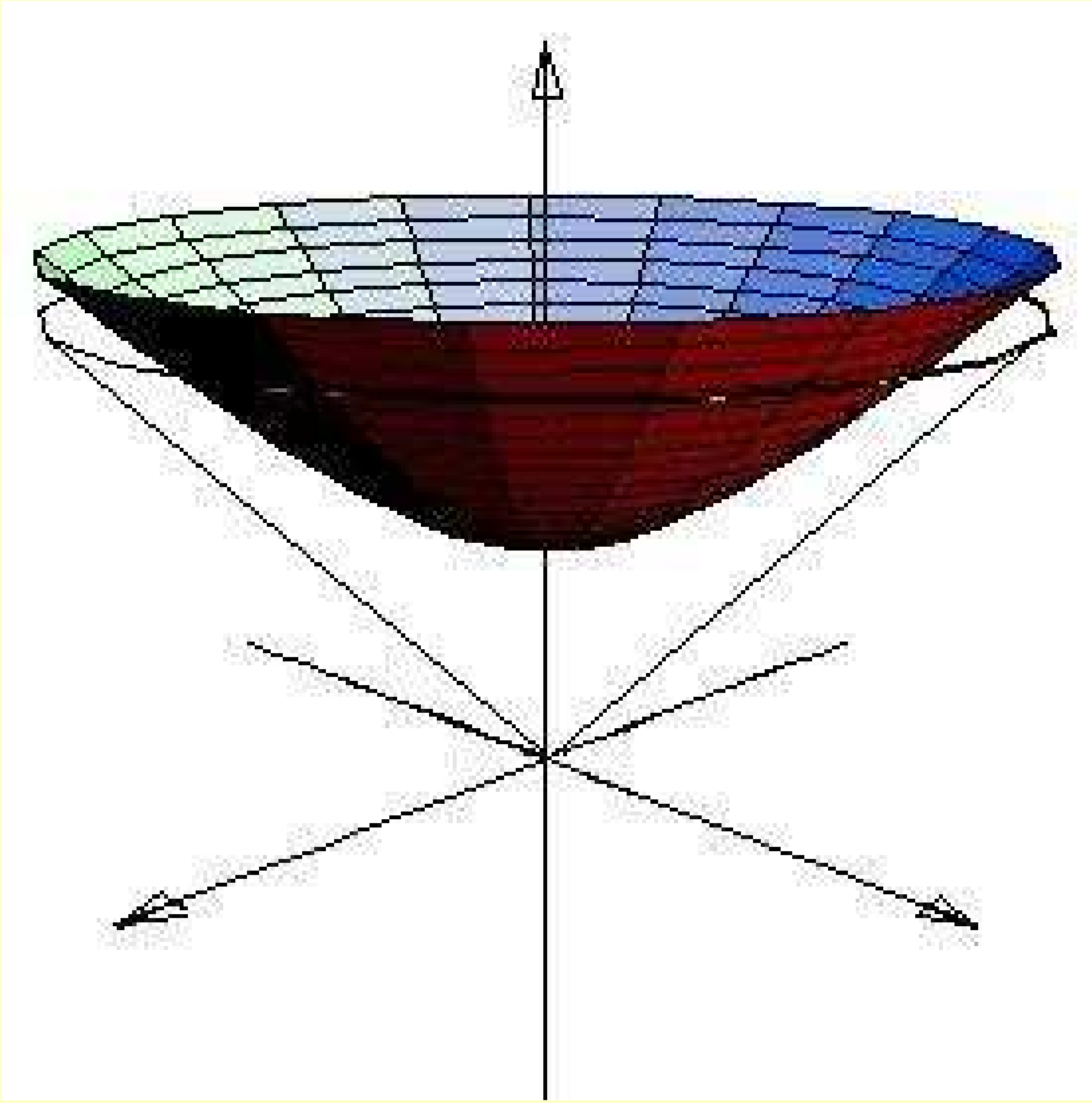
$$R^2 = X_0^2 - \sum_{i=1}^{5} X_i^2$$

Poincaré patch of $EAdS_5$ **:**

$$z = \frac{R^2}{X_0 + X_5} \ (z > 0), \qquad x_\mu = \frac{z X_\mu}{R} \ (\mu = 1, \cdots 4)$$

$$ds^2 = \frac{R^2}{z^2}\left(dz^2 + dx^2\right)$$

Geometry of $EAdS_2$



Duality

IIB superstring in $EAdS_5 \simeq$
$SU(N)$ **super Yang-Mills on** $\partial EAdS_5 \simeq \mathbb{R}^4$ **.**

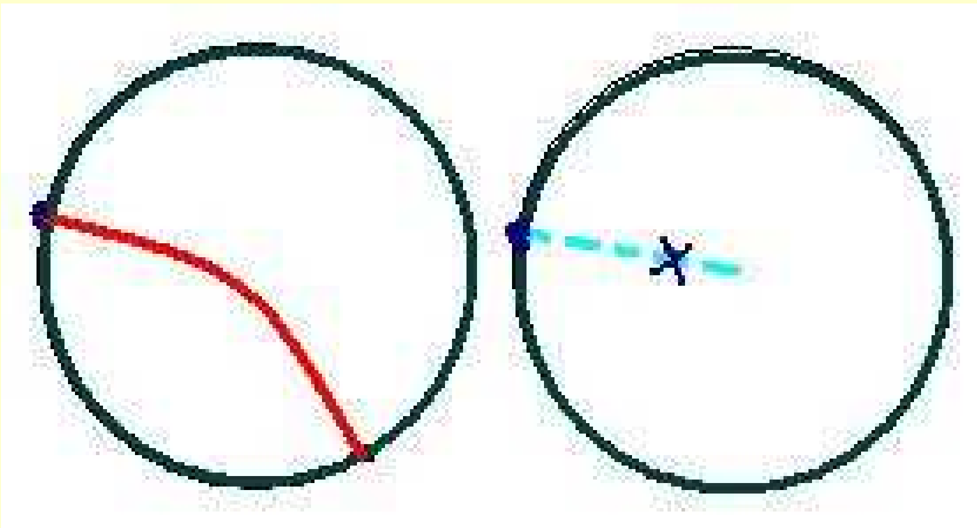
Isometry of $EAdS_5 \simeq SO(5,1)$
$\simeq$ **conformal symmetry of** $\mathbb{R}^4$**.**

YM instanton and its dual are good probes.

What is the dual of YM instanton?

Possibilities:

- 1D object with an endpoint on the boundary
  This works only for AdS_5 .
  or
- 0D object in $EAdS_5$
  How to match the coordinates?



Supergravity (Einstein frame)

$$R_{\mu\nu} - \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}e^{2\phi}\partial_\mu\chi\partial_\nu\chi = \frac{1}{6}F_{\alpha\beta\gamma\lambda\mu}F^{\alpha\beta\gamma\lambda}{}_\nu$$

$$\nabla_\mu\left(e^{2\phi}\partial^\mu\chi\right) = 0 \qquad \nabla^2\phi + e^{2\phi}\partial_\mu\chi\partial^\mu\chi = 0$$

$\phi =$ **dilaton,** $\chi =$ **axion**

Background: $EAdS_5$

$$\begin{cases} \phi = 0 = \chi \\ F = \textbf{self-dual vol. form} \rightarrow FF \propto g. \end{cases}$$

“Self-dual” ansatz

$$(\tau - \tau_\infty) = \pm i(\tau - \tau_\infty)^*$$

$$\tau \equiv \chi + ie^{-\phi}$$

$$\chi - \chi_\infty = \pm\left(e^{-\phi} - e^{-\phi_\infty}\right)$$

Supergravity equations become

$$R_{\mu\nu} = \frac{1}{6} F_{\alpha\beta\gamma\lambda\mu} F^{\alpha\beta\gamma\lambda}{}_{\nu}$$

$$\nabla^2 e^{\phi} = 0$$

S-duality

$$\tau \to \tau' = \frac{\alpha\tau + \beta\tau}{\gamma\tau + \delta\tau} \qquad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, \mathbb{R})$$

The "self-dual" ansatz is very special:

- The background metric is not affected.
- It is self-dual like.
- It preserves 1/2 SUSY.
- The action becomes

$$S = -\int d^5x \;\; \partial_\mu \left(e^{2\phi} \chi g^{\mu\nu} \partial_\nu \chi \right)$$

$$= \frac{2\pi n}{g_\infty} \qquad (g_\infty = e^{\phi_\infty})$$

D-instanton solution

Chu, Ho, Wu: hep-th/9806103

$$e^{\phi} = c_0 + c_1 \frac{(d^2+2)(d^4+4d^2-2)}{d^3(d^2+4)^{3/2}}$$

$d = SO(1,5)$-**invariant length**

$$d^2 \equiv \frac{\eta^{AB} X_A X_B}{R^2} = \frac{1}{z_0 z}\left[(z-z_0)^2 + |x-x_0|^2\right]$$

$(z_0, x_0) =$ **location of the D-instanton**

Approaching to the boundary ($z \sim 0$)

$$e^{\phi} \simeq e^{\phi_\infty} - 6c_1 z^4 Z^4(x; x_0, z_0)$$

$$\chi \simeq \chi_\infty \pm 6c_1 e^{-2\phi_\infty} z^4 Z^4(x; x_0, z_0)$$

$$Z(x; x_0, z_0) \equiv \frac{z_0}{z_0^2 + |x - x_0|^2}$$

Déjà vu!

D instanton = YM instanton

$$L = z_0 \qquad c_1 = \frac{4\pi n}{N^2} \qquad n = Q$$

Charge quantization due to QM

$$e^{-\phi} - e^{-\phi_\infty} \leftrightarrow \textbf{Tr}\, F_{\mu\nu} F^{\mu\nu}$$

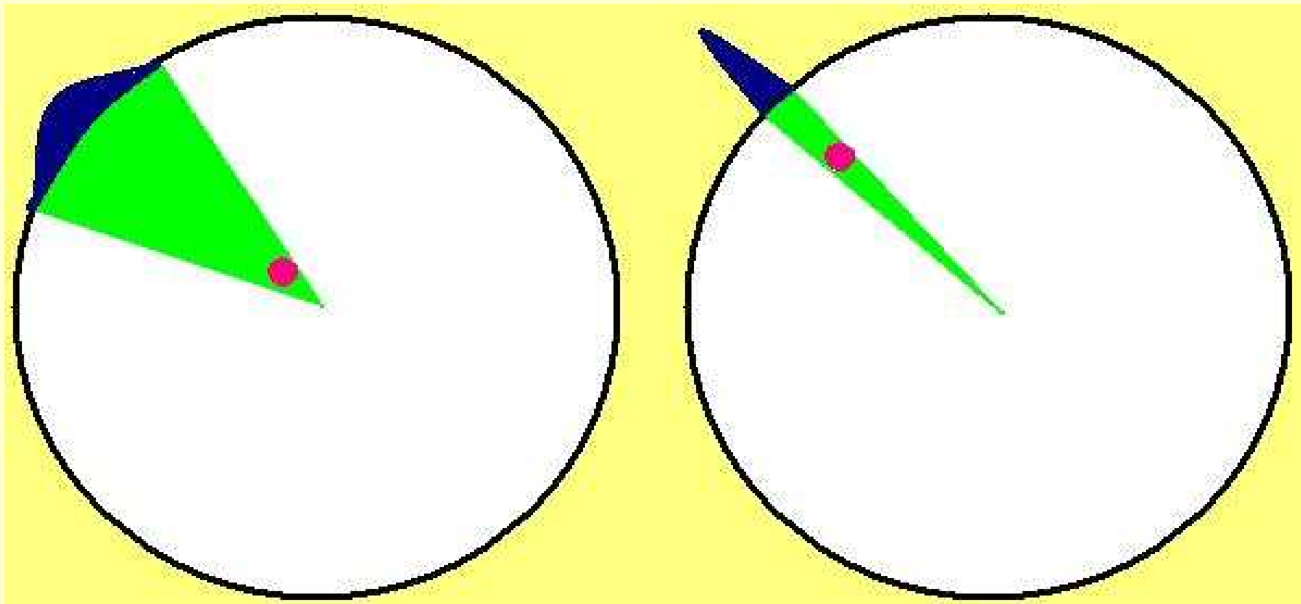
$$\chi - \chi_\infty \leftrightarrow \textbf{Tr}\, F_{\mu\nu}{}^* F^{\mu\nu}$$

Scaling $(z, x) \to (\lambda z, \lambda x)$ **is a symmetry of the** $EAdS_5$ **metric** $ds^2 = \frac{R^2}{z^2}(dz^2 + dx^2)$.

$\mathbb{R}^4$ **:** $(L, x) \to (\lambda L, \lambda x)$ **preserves the YM instanton sol.** $A = \left(\frac{r^2}{r^2+L^2}\right) U^{-1} dU$

where $U = \frac{x^4 - i x^a \sigma_a}{r}$.

$$\Rightarrow \quad z_0 = L$$



Generalizations:

- **B-field background** $\longrightarrow$ **NC** $\mathbb{R}^4$

- $EAdS_5$ **w. black hole** $\longrightarrow \mathbb{R}^3 \times S^1$

Rey, Jun. 05

- $EAdS_5$ $\longrightarrow$ **conformally compact Einstein manifolds.**

ADHM multi-instantons

For $\mathcal{N} = 4$ $SU(N)$ super YM theory:

The $N \gg k$ limit of k instanton moduli space has the geometry of a single copy of $EAdS_5 \times S^5$. (Instantons tend to stay on top of each other.)

The $SU(2)$ embeddings in $SU(N)$ are mutually commuting.

Dorey, Hollowood, Khoze, Mattis, Vandoren: hep-th/9901128

Dorey, Hollowood, Khoze, Mattis: hep-th/0206063

For finite N:

Corrections at the subleading term in the $1/N$ expansion

Something must happen when

$$k = N/2 \to N/2 + 1.$$

Supergravity approximation of superstring is good only for large N.

Quantum effect is important for finite N.

Quantum deformation of $EAdS_5$

q-$EAdS_5$

Jevicki, Ramgoolam: hep-th/9902059; Ho, Ramgoolam, Tatar: hep-th/9907145; Ho, Li: hep-th/0004072, hep-th/0005268.

q-AdS_2: Lie algebra of $SL(2,\mathbb{R}) \simeq$ algebra of the Cartesian coordinates.

q-AdS_2 can be defined by the representation (j,α) with $j = 1/2 + iN$ and α is the VEV of χ.

Noncommutative instanton

Nekrasov, Schwarz: hep-th/9802068

ADHM construction

$$V = \mathbb{C}^N, \qquad W = \mathbb{C}^k$$

$$B_0, B_1 \in Hom(V,V), \qquad I, J^\dagger \in Hom(W,V)$$

$$\mu_r \equiv [B_0, B_0^\dagger] + [B_1, B_1^\dagger] + II^\dagger - J^\dagger J$$

$$\mu_c \equiv [B_0, B_1] + IJ$$

Instantons on $\mathbb{R}^4$ are parametrized by solutions of $(\mu_r = 0,\ \mu_c = 0)$ modulo equivalence relations.

$$\mathcal{M} = \left(\mu_r^{-1}(0) + \mu_c^{-1}(0)\right)/U(V)$$

$\mathcal{M}$ has conical singularities.

Noncommutative $\mathbb{R}^4$ is defined by

$$[z_0, \bar{z}_0] = [z_1, \bar{z}_1] = -\frac{\zeta}{2}$$

ADHM on noncommutative $\mathbb{R}^4$

$$\mu_r = \zeta, \qquad \mu_c = 0$$

$$\mathcal{M} = \left(\mu_r^{-1}(\zeta) + \mu_c^{-1}(0)\right)/U(V)$$

Singularities resolved by the deformation.

(Deformation of $\mu_c = 0$ is equivalent.)

Conclusion

- **D-instanton $\leftrightarrow$ YM instanton**
- **isometry $\leftrightarrow$ conformal symemtry**
- **Moduli space for $k = 1$ and large N**
- **AdS/CFT $\rightarrow$ notion of (q-AdS_5)$_N$**
- **Generalizations**