Game Theory: Lecture 4

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Outline

- Information process-the partition model
  - Common knowledge
  - Agreeing to disagree
- Epistemic conditions for solution concepts
  - Correlated equilibrium and Nash equilibrium
  - Rationalizability

1 The Partition Model

1.1 Information partition and knowledge operator

Let Ω be a finite set of *states*. A model of *information structure* for n agents consists of a common prior $\mu \in \Delta(\Omega)$ and a *knowledge operator* $K_i : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$ for each agent $i = 1, ..., n$. K_i is a *knowledge operator* if it satisfies the following properties:

(K1) $K_i(E) \subseteq E$.

(K2) $E \subseteq F$ implies that $K_i(E) \subseteq K_i(F)$.

(K3) $\sim K_i(E) \subseteq K_i(\sim K_i(E))$.

Given the knowledge operator K_i, define the *knowledge ufield* $\mathcal{K}_i$ as

$$\mathcal{K}_i = \{K_i E : E \subseteq \Omega\}.$$

A different approach is to define *information partition* $I_i : \Omega \to \mathcal{P}(\Omega)$ such that

(I1) for all $\omega \in \Omega$, $\omega \in I_i(\omega)$;

(I2) for all $\omega, \omega' \in \Omega$, either $I_i(\omega) = I_i(\omega')$ or $I_i(\omega) \cap I_i(\omega') = \emptyset$.

In this approach, the knowledge ufield is defined as

$$\mathcal{K}_i = \{\bigcup_{\omega \in A} I(\omega) : A \subseteq \Omega\}.$$

These two approaches are equivalent in the following sense: one can either take I_i as primitive and define $K_i(E) = \{\omega : I(\omega) \subseteq E\}$, or take K_i as primitive and define $I_i(\omega) = \sim K_i(\sim \{\omega\})$.

Lemma 1.1. *(1) Let I_i be an information partition. Define $K_i(E) = \{\omega : I(\omega) \subseteq E\}$. Then K_i satisfies (K1-K3).*

(2) Let K_i be a knowledge operator. Define $I_i(\omega) = \sim K_i(\sim \{\omega\})$. Then I_i satisfies (I1-I2) and $K_i(E) = \{\omega : I_i(\omega) \subset E\}$ for any event E.

Proof. (1) (K1) $\omega \in K_i(E)$ only if $\omega \in I_i(\omega) \subseteq E$.
(K2) $\omega \in K_i(E)$ only if $\omega \in I_i(\omega) \subseteq E \subseteq F$ and so $\omega \in K_i(F)$.
(K3) Suppose that $\omega \notin K_i(E)$. Then $I_i(\omega) \cap \sim E \neq \emptyset$. Hence, for all $\omega' \in I_i(\omega)$, $\omega' \notin K_i(E)$, and so $I_i(\omega) \subseteq \sim K_i(E)$. Thus, $\omega \in K_i(\sim K_i(E))$.

(2) (I1) By (K1), $K_i(\sim \{\omega\}) \subseteq \sim \{\omega\}$ and so $\omega \in \sim K_i(\sim \{\omega\}) = I_i(\omega)$.

(I2) Define $\mathcal{K}_i = \{K_i E : E \subseteq \Omega\}$. Then we have the following two claims: (a) $I_i(\omega) \in \mathcal{K}_i$; (b) $I_i(\omega) = \bigcap \{F \in \mathcal{K}_i : \omega \in F\}$.

(a) By (K1) and (K3), $I_i(\omega) = K_i(\sim K_i(\sim \{\omega\}))$ and hence $I_i(\omega) \in \mathcal{K}_i$.

(b) Suppose that $\omega \in F \in \mathcal{K}_i$. Then $F = K_iE$ for some E. So $\sim K_iE =\sim F \subseteq\sim \{\omega\}$. By (K3) and (K2), $\sim K_i(E) = K_i(\sim K_i(E)) \subseteq K_i(\sim \{\omega\})$. Then, $I_i(\omega) =\sim K_i(\sim \{\omega\}) \subseteq K_i(E) = F$. Thus, $I_i(\omega) \subseteq \bigcap\{F \in \mathcal{K}_i : \omega \in F\}$. On the other hand, by (I1) and (a), $\omega \in I_i(\omega) \in \mathcal{K}_i$ and hence $\bigcap\{F \in \mathcal{K}_i : \omega \in F\} \subseteq I_i(\omega)$.

Let $\omega, \omega' \in \Omega$. To show (I2), consider two cases: (i) for each $F \in \mathcal{K}_i$, $\omega \in F$ if and only if $\omega' \in F$; (ii) for some $F \in \mathcal{K}_i$, $\omega \in F$ and $\omega' \notin F$. If (i) is true, then by claim (b) above, $I_i(\omega) = I_i(\omega')$. If case (ii) is true, then $I_i(\omega) \cap I_i(\omega') = \emptyset$ because $\sim F \in \mathcal{K}_i$ as well by (K1) and (K3).

Finally, we show that $K_i(E) = \{\omega \in \Omega : I_i(\omega) \subseteq E\}$. First assume that $I_i(\omega) \subseteq E$. By (K2) $K_i(I_i(\omega)) \subseteq K_i(E)$ and by (K3) $I_i(\omega) \subseteq K_i(I_i(\omega))$ and hence $I_i(\omega) \subseteq K_i(E)$. On the other hand, if $\omega \in K_i(E)$, then by claim (b) above $I_i(\omega) \subseteq K_i(E)$. □

Lemma 1.2. *Suppose that K_i satisfies (K1-K3). Then*
(1) for any collection of events $\{E_\alpha : \alpha \in \mathcal{A}\}$, $\bigcap_{\alpha\in\mathcal{A}} K_i(E_\alpha) = K_i(\bigcap_{\alpha\in\mathcal{A}} E_\alpha)$;
(2) for any event E, $K_i(K_i(E)) = K_i(E)$.

Proof. (1) Because $\bigcap_{\alpha\in\mathcal{A}} E_\alpha \subseteq E_\alpha$, (K2) implies that $K_i(\bigcap_{\alpha\in\mathcal{A}} E_\alpha) \subseteq \bigcap_{\alpha\in\mathcal{A}} K_i(E_\alpha)$.

On the other hand, if $\omega \in \bigcap_{\alpha\in\mathcal{A}} K_i(E_\alpha)$, then $I_i(\omega) \subseteq E_\alpha$ for all α, and hence $\omega \in K_i(\bigcap_{\alpha\in\mathcal{A}} E_\alpha)$.

(2) By (K1) and (K3) $\sim K_i(E) = K_i(\sim K_i(E))$ and hence $K_i(E) =\sim K_i(\sim K_i(E))$. Therefore, $K_i(K_i(E)) = K_i(\sim K_i(\sim K_i(E))) =\sim K_i(\sim K_i(E)) = K_i(E)$. □

1.2 Common knowledge

Now we turn to common knowledge. Define $K^m : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$ inductively as follows: $K^1(E) = \bigcap_{i=1}^n K_i(E)$; for $m > 1$, $K^m(E) = \bigcap_{i=1}^n K_i(K^{m-1}(E))$. K^m is the mth order knowledge operator. The *common knowledge operator*, CK, is defined as $CK(E) = \bigcap_{m\in\mathbb{N}} K^m(E)$.

Lemma 1.3. *For each i, $K_i(CK(E)) = CK(E)$. Moreover, $CK(E) \subseteq E$ and, if $E \subseteq F$, $CK(E) \subseteq CK(F)$.*

Proof. By (K1) $K_i(CK(E)) \subseteq CK(E)$. On the other hand, by Lemma 1.2,

$$CK(E) = \bigcap_{m \in \mathbb{N}} K^m(E) \subseteq \bigcap_{m \geq 2} K^m(E)$$

$$= \bigcap_{m \in \mathbb{N}} K^1(K^m(E)) \subseteq \bigcap_{m \in \mathbb{N}} K_i(K^m(E)) = K_i(\bigcap_{m \in \mathbb{N}} K^m(E)) = K_i(CK(E)).$$

$K_i(E) \subseteq E$ for all i and hence $CK(E) \subseteq K_i(E) \subseteq E$. Suppose that $E \subseteq F$. Then for each m, $K^m(E) \subseteq K^m(F)$ for all $m \in \mathbb{N}$. Hence $CK(E) \subseteq CK(F)$. □

An event F is self-evident if $F \subseteq K_i(F)$ for all $i = 1, ..., n$.

Lemma 1.4. *An event F is self-evident if and only if $F \in \bigcap_{i=1,\ldots,n} \mathcal{K}_i$.*

Proof. Suppose that F is self-evident. Then by (K1) $F = K_i(F)$ for all $i = 1, ..., n$ and hence $F \in \bigcap_{i=1,\ldots,n} \mathcal{K}_i$. Conversely, if $F \in \bigcap_{i=1,\ldots,n} \mathcal{K}_i$, then $F = K_i(F)$ for each $i = 1, ..., n$ and so F is self-evident. □

Self-evidence is closely related to common knowledge: Lemma 1.3 shows that the event $CK(E)$ is self-evident; the following theorem shows that the event $CK(E)$ is the largest self-evident event included in E. Let $\mathcal{K} = \bigcap_{i=1,\ldots,n} \mathcal{K}_i$.

Theorem 1.1. *$CK(E)$ is the largest self-evident subset of E.*

Proof. Suppose that $F \subseteq E$ and $F \in \mathcal{K}$. Then $F \subseteq K^1(E)$. We show, by induction, that $F \subseteq K^m(E)$. Suppose that $F \subseteq K^m(E)$. Then by (K2), $F = K_i(F) \subseteq K_i(K^m(E))$ for each $i = 1, ..., n$. Thus, $F \subseteq K^{m+1}(E)$. Therefore, $F \subseteq CK(E)$. On the other hand, $CK(E)$ is also self-evident. □

For each $i = 1, ..., n$, $\mathcal{K}_i$ is closed under union (Lemma 1.2) and complementation (by (K1) and (K3)) and K_i is defined by $K_i(E) = \bigcup\{F \in \mathcal{K}_i : F \subseteq E\}$. It is easy to show that this defines a knowledge operator. Now, $\mathcal{K}$ is also a universal field, i.e., it is closed under complementation and union, and hence CK is also a knowledge operator by Theorem 1.1.

1.3 Agreeing to Disagree

Consider information partitions $\{I_i : i = 1, ..., n\}$ on a state space Ω and a prior distribution $\mu \in \Delta(\Omega)$. For any event E, agent i's a posterior probability on E at state ω is $\mu(E|I_i(\omega))$.

Theorem 1.2 (Aumann (1976))**.** *Let $E \subseteq \Omega$ be an event. If it is common knowledge at ω that $\mu(E|I_i(\omega)) = p_i$ for all $i = 1, .., n$, then $p_1 = p_2 = ... = p_n$.*

Proof. Let $F = \{\omega : \mu(E|I_i(\omega)) = p_i, i = 1, ..., n\}$ and let $F^* = CK(F)$. Then $F^* \subseteq F$ and $F^* = K_i(F^*)$ for all i. For each i, $F^* = \bigcup\{I_i(\omega) : \omega \in F^*\}$. Fix some i. Because $F^* \subseteq F$, for each $\omega \in F^*$, $\mu(E|I_i(\omega)) = p_i$. Thus,

$$\mu(E|\bigcup\{I_i(\omega) : \omega \in F^*\}) = p_i \text{ and so } \mu(E|F^*) = p_i.$$

Therefore, $p_1 = p_2 = ... = p_n$. □

1.4 The Harsanyi's doctrine and the CPA

In the previous sections players' priors are exogenously given. Savage (1954) formulates a axiomatic framework for subjective probability. However, that approach only asserts that a coherent decision-making in the face of uncertainty requires the use of a probability distribution, but does not provide a theory about how such a probability is derived. Nonetheless, for a given prior distribution, the Savage theory does give a procedure for updating the probabilistic belief when new information arrives, namely, the Bayes' rule.

Harsanyi supplements the gap by asserting the existence of a universal prior before any information. That universal prior is the *common prior*. A corollary of this doctrine is that difference in probabilistic beliefs reflects difference in information only.

2 Epistemic Conditions

2.1 Correlated Equilibrium

Let $G = \langle \{S_i\}_{i=1}^n, \{u_i\}_{i=1}^n \rangle$ be a finite n-person game. Consider an information structure $\langle \Omega, \{I_i\}_{i=1}^n, \mu \rangle$. A strategy for agent i is a function $\sigma_i : \Omega \to S_i$ that is measurable w.r.t. I_i. A correlated equilibrium is a list of strategies $(\sigma_1, ...\sigma_n)$ such that for each $i = 1, .., n$, and for each strategy τ_i,

$$\sum_{\omega \in \Omega} \mu(\omega) u_i(\sigma_i(\omega); \sigma_{-i}(\omega)) \geq \sum_{\omega \in \Omega} \mu(\omega) u_i(\tau_i(\omega); \sigma_{-i}(\omega)).$$

A correlated equilibrium σ generates a distribution $\pi \in \Delta(S_1 \times .... \times S_n)$ as follows: for each $s \in S = S_1 \times S_2 \times ... \times S_n$,

$$\pi(s) = \mu(\{\omega \in \Omega : (\forall i = 1, .., n) \sigma_i(\omega) = s_i\}).$$

Lemma 2.1. *A distribution of actions $\pi \in \Delta(S)$ is generated by a correlated equilibrium if and only if for each $i = 1, ..., n$, and for each s_i such that $\pi(s_i) > 0$ and for each $s'_i \in S_i$,*

$$\sum_{s_{-i} \in S_{-i}} \pi(s_{-i}|s_i) u(s_i, s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} \pi(s_{-i}|s_i) u(s'_i, s_{-i}).$$

A easy corollary of this lemma is that any Nash equilibrium in mixed strategies is also a correlated equilibrium. In fact, in terms of payoffs, correlated equilibria correspond to the convex haul of Nash equilibria.

Lemma 2.2. *Suppose that π^1 and π^2 are two distributions on actions that are generated by some correlated equilibria of G. Then $\alpha\pi^1 + (1 - \alpha)\pi^2$ is also a distribution generated by some correlated equilibrium of G.*

Aumann (1987) shows that correlated equilibrium is consistent with common knowledge of rationality, in terms of expected utility maximization. Let $\langle \Omega, \{I_i\}_{i=1}^n, \mu \rangle$ be an information structure. Let $\mathbf{s}_i : \Omega \to S_i$ be the function that assigns an action at each state. Agent i is said to be *rational* at state ω if $\mathbf{s}_i(\omega)$ maximizes his expected utility against his posterior belief on Ω. The common prior μ and the assignments $\{\mathbf{s}_i\}_{i=1}^n$ generate a distribution $\pi \in \Delta(S)$ given by

$$\pi(s) = \mu(\{\omega \in \Omega : (\forall i = 1, ..., n)\mathbf{s}_i(\omega) = s_i\}.$$

Theorem 2.1 (Aumann, 1987)**.** *Suppose that each agent is rational at every state. Then the generated distribution π from the state space is a distribution generated by some correlated equilibrium.*

2.2 Type space and Nash equilibrium

Now we turn to Nash equilibrium. To study this we also introduce *type spaces*. A type space is a list $\langle \{T_i\}_{i=1}^n, \{\mu_i\}_{i=1}^n \rangle$ in which for each $i = 1, ..., n$, $\mu_i : T_i \to \Delta(\prod_{j \neq i} T_j)$. We say that there is a common prior μ if for each i and for each $t_i \in T_i$, $\mu_i(t_i)(t_{-i}) = \mu(t_{-i}|t_i)$ whenever $\mu[t_i] > 0$. For any event $E \subseteq T$, define

$$B_i(E) = \{t \in T : \mu_i(t_i)(E^{t_i}) = 1\},$$

where $E^{t_i} = \{t_{-i} \in T_{-i} : (t_i, t_{-i}) \in E\}$. Then we define B^m inductively as follows: $B^1(E) = \bigcap_{i=1}^n B_i(E)$; for $m > 1$, $B^m(E) = \bigcap_{i=1}^n B_i(B^{m-1}(E))$. The common belief operator CB is defined as $CB(E) = \bigcap_{m=1}^\infty B^m(E)$.

Given a game $G = \langle \{S_i\}_{i=1}^n, \{u_i\}_{i=1}^n \rangle$, let $\mathbf{s}_i : T_i \to S_i$ be a function that assigns each type t_i an action. Given the assignments $\{\mathbf{s}_i\}_{i=1}^n$, agent i's conjecture at state t is $\phi_i(t) \in \Delta(S_{-i})$ given by

$$\phi_i(t_i)(s_{-i}) = \mu_i(t_i)(\{t_{-i} \in T_{-i} : \mathbf{s}_{-i}(t_{-i}) = s_{-i}\}).$$

Agent i is *rational* at state t if $\mathbf{s}_i(t)$ maximizes the expected payoffs against the conjecture $\phi_i(t)$.

Lemma 2.3. *Let $s \in S$. Suppose that at state t, it is mutual knowledge that all agents are rational, and the action profile is s. Then s is a Nash equilibrium.*

Theorem 2.2 (Aumann and Brandenburger, 1995, Theorem A)**.** *Suppose that $n = 2$. Let $\varphi_i \in \Delta(S_{-i})$, $i = 1, 2$, be two conjectures. Suppose that at state t^*, it is mutually believed* *that both agents are rational, and that the conjectures are given by (φ_1, φ_2). Then, (φ_2, φ_1) is a Nash equilibrium.*

Theorem 2.3 (Aumann and Brandenburger, 1995, Theorem B)**.** *Let $\varphi_i \in \Delta(S_{-i})$, $i = 1, ..., n$, be a profile of conjectures. Suppose that agents have a common prior $\mu \in \Delta(\Omega)$, which assigns positive probability to it being mutually believed that both agents are rational, and commonly believed that the conjectures are given by $(\varphi_1, ..., \varphi_n)$. Then, for all j, all conjectures φ_i with $j \neq i$ induce the same conjecture σ_j about j, and $(\sigma_1, ..., \sigma_n)$ is a Nash equilibrium.*

2.3 Rationalizability

Rationalizable set

Consider a normal-form game with complete information:

$$G := (I, \{S_i\}_{i \in I}, \{u_i\}_{i \in I}),$$

where I denotes a finite set of players, S_i is the strategy set for player i (as usual, $S = \prod_{i \in I} S_i$), and $u_i : S \to \Re$ denotes the payoff function for player i.

Definition 2.1. The (correlated) **rationalizable set** $R(G)$ is defined as $\bigcap_{n \in \mathbb{N}} \Lambda^n(S)$, where $\Lambda_i(S') := \{s_i \in S_i : \text{there exists } \mu \in \Delta(S_{-i}) \text{ such that } \mu(S'_{-i}) = 1 \text{ and } s_i \in BR_i(\mu)\}$ for any Borel subset $S' = \prod_{i \in I} S'_i \subseteq S$ and $\Lambda(S') := \prod_{i \in I} \Lambda_i(S')$; $\Lambda^0(S) := \Lambda(S)$, and $\Lambda^{n+1}(S) := \Lambda(\Lambda^n(S))$ is defined inductively for $n \geq 0$. .

Theorem 2.4. *Suppose that in game G, S is compact.*

(a) $\Lambda^{n+1}(S) \subseteq \Lambda^n(S)$, for all $n \geq 0$.

(b) $\Lambda(R(G)) = R(G)$.

Proof. (a) When $n = 0$, since $s_i \in \Lambda_i^{p,1}(S)$ implies that for some μ_i, $s_i \in BR_i(\mu_i)$, $s_i \in \Lambda_i^{p,0}(S) = \Lambda_i^1(S)$. Suppose that $\Lambda^{p,k+1}(S) \subseteq \Lambda^{p,k}(S)$, for all $k < n$. Let $s_i \in \Lambda^{p,n+1}$. Then there exists μ_i such that $\mu_i(\Lambda_{-i}^{p,n}) \geq p$ and $s_i \in BR_i(\mu_i)$. By induction hypothesis, $\Lambda^{p,n}(S) \subseteq \Lambda^{p,n-1}(S)$, and so $\mu_i(\Lambda^{p,n-1}(S)) \geq \mu_i(\Lambda^{p,n}(S)) \geq p$. Thus, $s_i \in \Lambda_i^{p,n}(S)$.

(b) Let $s_i \in \Lambda_i^p(R^p(G))$. Then, there exists μ_i such that $\mu_i(R_{-i}^p(G)) \geq p$ and $s_i \in BR_i(\mu_i)$. Since for each n, $R^p(G) \subseteq \Lambda^{p,n}(S)$, $\mu_i(\Lambda_{-i}^{p,n}(S)) \geq p$. Thus, $s_i \in R^p(G)$.

Conversely, let $s_i \in R^p(G)$. Thus, for each n, there exists μ_i^n such that $\mu_i^n(\Lambda_{-i}^{p,n}(S)) \geq p$ and $s_i \in BR_i(\mu_i^n)$. Then, since $\Delta(S_{-i})$ is compact, there is a subsequence μ_i^k which converges (weakly) to μ_i. For each k_0, $\mu_i^k(\Lambda_{-i}^{p,k_0}(S)) \geq p$ for all $k > k_0$ by (a). Thus, $\mu_i(\Lambda_{-i}^{p,k_0}(S)) \geq \limsup \mu_i^k(\Lambda_{-i}^{p,k_0}(S))) \geq p$. Since $R^p(G) = \lim_n \Lambda^{p,n}(S)$, $\mu_i(R_{-i}^p(G)) = \lim_n \mu_i(\Lambda_{-i}^{k,n}(S)) \geq p$. Therefore, $s_i \in \Lambda_i^p(R^p(G))$. $\square$

Property (b) is known as the best-response-property, which is given by [11] for rationalizable sets.

Universal type space and epistemic conditions for rationalizability

For any topological space A, let $\Delta(A)$ be the set of all probability measures on the Borel subsets of A, equipped with the weak* topology. If the underlying space is compact and metric, then so is the set of all probability measures over its Borel sets.

Now we provide the hierarchies of beliefs associated with G. Let $X_i^0 := S_{-i}$ be the first-order uncertainty of player i and so $\Delta(S_{-i})$ is the set of i's first-order beliefs. Inductively define the nth-order uncertainties X_i^n as $X_i^n := X_i^{n-1} \times \prod_{j \neq i} \Delta(X_j^{n-1})$ and thus the set of i's nth-order beliefs is $\Delta(X_i^n)$, for all $n \geq 1$. It is implicitly assumed here that every player is certain of his own strategic choice and this fact is common certainty among players*.

By a type of player i we mean an element $t_i \in T_i^0 := \prod_{n=0}^{\infty} \Delta(X_i^n)$, which describes player i's beliefs over all possible uncertainties. A type $t_i = (\delta_i^0, \delta_i^1, ...)$ is coherent if for all $n \in \mathbb{N}$, $\operatorname{marg}_{X_i^{n-1}} \delta_i^n = \delta_i^{n-1}$. By coherence we mean that any lower-order belief of a type can be derived from the beliefs of higher orders. Merterns and Zamir (1985) shows that there is a homeomorphism $\varphi_i : T_i^1 \to \Delta(S_{-i} \times T_{-i}^0)$ for each $i \in I$, where T_i^1 is the set of coherent types. Then define $T_i^n := \{t_i \in T_i^1 : \varphi_i(t_i)(S_{-i} \times T_{-i}^{n-1}) = 1\}$ inductively for all $n \geq 2$. Let $T_i = \bigcap_{n=1}^{\infty} T_i^n$. Then φ_i restricted to T_i is a homeomorphism between

*One may, however, construct a universal type space with S as the first-order uncertainty and then assume that it is common certainty that every one is certain of his own strategy, but the two approaches are essentially equivalent.

T_i and $\Delta(S_{-i} \times T_{-i})$ such that $\text{marg}_{X_i^n} \varphi_i(t_i) = \delta_i^n$ for all $n \geq 0$, where $t_i = (\delta_i^0, \delta_i^1, ...)$. The existence of a homeomorphism is regarded as the universality of the universal type space T.

One can also characterize the set T_i in terms of finite-order beliefs. Let $C_i^1 := \{t_i \in T_i^0 : \text{marg}_{X_i^0} \delta_i^1 = \delta_i^0\}$. The set C_i^1 consists of the types for which player i's second-order and first-order beliefs are coherent. Define $C_i^n := \{t_i \in C_i^{n-1} : \text{marg}_{X_i^n} \delta_i^n = \delta_i^{n-1}$ and $\delta_i^n(\text{Proj}_{X_i^n} S_{-i} \times C_{-i}^{n-1}) = 1\}$ inductively for all $n \geq 2$. The set C_i^n then consists of the types where player i is certain that other players' beliefs up to the nth-order are coherent and his own beliefs up to the $n+1$th order are coherent. By construction, $\text{Proj}_{\Delta(X_i^0) \times \ldots \times \Delta(X_i^n)} C_i^{n+1} = C_i^n$ for all $n > 0$. The following lemma is the formal characterization.

Lemma 2.4. *For all $i \in I$, $T_i = \bigcap_{n=1}^{\infty} C_i^n$.*

Proof. By comparing the definitions, we have $T_i^n \subseteq C_i^n$. We show that $\bigcap_{n=1}^{\infty} C_i^n \subseteq T_i$ for all $i \in I$ in the following. Let $t_i \in \bigcap_{n=1}^{\infty} C_i^n$. Clearly, $t_i \in T_i^1$. We shall show that $\varphi_i(t_i)((\text{Proj}_{X_i^n} S_{-i} \times T_{-i}^1) \times \prod_{k=n}^{\infty}(\prod_{j \neq i} \Delta(X_j^k))) = 1$ for all n by mathematical induction.

Now we have $\text{Proj}_{\Delta(X_i^0) \times \ldots \times \Delta(X_i^n)} C_i^n \subseteq \text{Proj}_{\Delta(X_i^0) \times \ldots \times \Delta(X_i^n)} T_i^1$ for all $n \in \mathbb{N}$. Since $t_i \in C_i^{n+1}$, $\varphi_i(t_i)((\text{Proj}_{X_i^n} S_{-i} \times T_{-i}^1) \times \prod_{k=n}^{\infty}(\prod_{j \neq i} \Delta(X_j^k))) \geq \varphi_i(t_i)((\text{Proj}_{X_i^n} S_{-i} \times C_{-i}^n) \times \prod_{k=n}^{\infty}(\prod_{j \neq i} \Delta(X_j^k))) = 1$. Thus, by continuity of probability measures, $\varphi_i(t_i)(S_{-i} \times T_{-i}^1) = 1$, i.e., $t_i \in T_i^2$. We show by induction that $t_i \in T_i^m$ for all m. Suppose that $\bigcap_{n=1}^{\infty} C_i^n \subseteq T_i^m$ for some $m \geq 2$. It follows that $\text{Proj}_{\Delta(X_i^0) \times \ldots \times \Delta(X_i^n)} C_i^n \subseteq \text{Proj}_{\Delta(X_i^0) \times \ldots \times \Delta(X_i^n)} T_i^m$ for all $n \in \mathbb{N}$. If $t_i \in \bigcap_{n=1}^{\infty} C_i^n$, then $\varphi_i(t_i)((\text{Proj}_{X_i^n} S_{-i} \times T_{-i}^m) \times \prod_{k=n}^{\infty}(\prod_{j \neq i} \Delta(X_j^k))) \geq \varphi_i(t_i)((\text{Proj}_{X_i^n} S_{-i} \times C_{-i}^n) \times \prod_{k=n}^{\infty}(\prod_{j \neq i} \Delta(X_j^k))) = 1$ for all n. Thus, $t_i \in T_i^{m+1}$. $\square$

This lemma is convenient in constructing a type from finite-order beliefs, and it will be used in our main result.

Define $\Omega := \prod_{i \in I}(S_i \times T_i)$ to be the state space. Each state describes the beliefs and strategic choices of all players, and the state space is a full description of all possible situations. In what follows, we use also s_i and t_i to denote the projections from Ω into

S_i and T_i, respectively. A subset of Ω is then regarded as a proposition for the situation, and is called an event. However, since this paper is concerned with probabilistic beliefs, events are confined to be measurable subsets with respect to the Borel σ-algebra on Ω.

In the state space, we can formulate rationality, certainty, common certainty, p-belief, and common p-belief in an explicit manner. A player is said to be rational in a game situation if the player chooses a strategy that maximizes the expected utility with respect to a probabilistic belief over the opponents' strategic choices.

A type for one player is certain of an event if the type assigns probability 1 over the event, and then the set of all types that have certainty over that is itself an event. We close this section by formally defining some notions that will be useful later.

◇ **Player i is rational** — $R_i := \{\omega \in \Omega : s_i(\omega) \in BR_i(\delta_i^0)$, with $t_i(\omega) = (\delta_i^0, ..., \delta_i^n, ...)\}$, where $BR_i(\mu) := \arg\max_{s_i \in S_i} \int_{S_{-i}} u_i(s)\, d\mu(s_{-i})$ for $\mu \in \Delta(S_{-i})$, i.e., the best response correspondence for player i.

◇ **Every player is rational** — $R := \bigcap_{i \in I} R_i$.

◇ The **belief operator** for player i — $B_i(E) := \{\omega \in \Omega : \varphi_i(t_i(\omega))(E^{\omega_i}) = 1\}$ for any event E on Ω, where $E^{\omega_i} := \{\omega_{-i} \in \Omega_{-i} : (\omega_i, \omega_{-i}) \in E\}$.

◇ The **mutual belief operator** — $BE := \bigcap_{i \in I} B_i E$, for any event E on Ω.

◇ For any event E on Ω, let $B^1 E := BE$, and then inductively define $B^{n+1}E := B(B^n E)$ for $n > 1$.

◇ The **common belief operator** — $CBE := \bigcap_{n \in \mathbb{N}} B^n E$ for any event E on Ω.

Theorem 2.5 (Tan and Werlang, 1988)**.** *A strategy profile s in G is rationalizable if and only if there exists a state $\omega = (s, t) \in CB(R)$.*