Game Theory: Lecture 3

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Outline

- Nash equilibrium
- Nash noncooperative theory
- *Ex ante* decision-making and decision criterion for Nash theory

1 Nash equilibrium

1.1 N-Person Normal Form Games

A N-person *normal form game* is given as a triple:

$$G = (N, \{S_i\}_{i \in N}, \{h_i\}_{i \in N}),$$

where

(1): $N = \{1, 2, ..., N\}$—the set of players;

(2): $S_i = \{\mathbf{s}_{i1}, ..., \mathbf{s}_{i\ell_i}\}$—the set of pure strategies for player $i = 1, 2, ..., N$;

(3): $h_i : S_1 \times S_2 \to \mathbb{R}$—the payoff function of player $i = 1, 2, ..., N$.

1.2 Existence

Let $\hat{G} = (N, \{\Delta(S_i)\}_{i\in N}, \{h_i\}_{i\in N})$ be the mixed extension of $G = (N, \{S_i\}_{i\in N}, \{h_i\}_{i\in N})$. A profile of mixed strategies $m^* = (m_1^*, ..., m_N^*)$ ($m_i^* \in \Delta(S_i)$ for $i \in N$) is called a *Nash equilibrium* iff for all $i \in N$,

$$h_i(m_i, m_{-i}^*) \leq h_i(m_i^*, m_{-i}^*) \text{ for all } m_i \in \Delta(S_i). \tag{1}$$

Here, we are using the notation: $m_{-i}^* = (m_1^*, ..., m_{i-1}^*, m_{i+1}^*, ..., m_N^*)$ and $(m_i, m_{-i}^*) = (m_1^*, ..., m_{i-1}^*, m_i, m_{i+1}^*, ..., m_N^*)$. Thus, (m_i^*, m_{-i}^*) is m^* itself.

The following is the famous theorem due to John F. Nash.

Theorem 1.1 (Nash (1951))**.** *Let $G = (N, \{S_i\}_{i\in N}, \{h_i\}_{i\in N})$ be a N-person finite normal form game. Then, the mixed extension $\hat{G} = (N, \{\Delta(S_i)\}_{i\in N}, \{h_i\}_{i\in N})$ has a Nash equilibrium.*

Let $G = (N, \{S_i\}_{i\in N}, \{h_i\}_{i\in N})$ be a 2-person zero-sum game. Then, the mixed extension $\hat{G}$ also satisfies the zero-sum condition. Since a Nash equilibrium becomes a saddle point, it follows from Theorem 1.1 that the mixed extension $\hat{G}$ has a saddle point.

Corollary 1.1. *Let $G = (N, \{S_i\}_{i\in N}, \{h_i\}_{i\in N})$ be an 2-person 0-sum game. Then, the mixed extension $\hat{G} = (N, \{\Delta(S_i)\}_{i\in N}, \{h_i\}_{i\in N})$ has a saddle point with respect to h_1.*

Moreover, because saddle point is equivalent to maximin strategies, and the existence of saddle points imply that the game is strictly determined, we have the Minimax theorem as a corollary of Theorem 1.1.

Theorem 1.2 (von Neumann (1928))**.** *Let $\hat{G}$ be the mixed extension of a 2-person 0-sum game G. Then,*

$$\max_{m_1\in M_1} \min_{m_2\in M_2} h_1(m_1, m_2) = \min_{m_2\in M_2} \max_{m_1\in M_1} h_1(m_1, m_2). \tag{2}$$

Theorem 1.1 is proved by applying Brouwer's fixed point theorem (or Kakutani's fixed point theorem). Now, we present Brouwer's fixed point theorem.

Let (R^m, d) be the m-dimensional Euclidean space with the Euclidean metric d, where

$$d(x, y) = \sqrt{\sum_{t=1}^{m} (x_t - y_t)^2} \ \text{ for } x, y \in R^m.$$

We say that a sequence $\{x^\nu\}$ *converges to* x^0 iff the sequence of real numbers $\{d(x^\nu, x^0)\}$ converges to 0.

Let T be a subset of R^m, i.e., $T \subseteq R^m$. We say that T is *closed* (in the topological sense) iff for any sequence $\{x^\nu\}$ in T, if $\{x^\nu\}$ converges to x^0 (in R^m), then x^0 belongs to T. Let T be a subset of R^m. We say that T is *bounded* iff there is a number M such that $d(0, x) \leq M$ for all $x \in T$.

We say that a subset T of R^m is *compact* iff T is closed and bounded. Hence, the interval $[0, 1]$ is compact, and the m-dimensional simplex is compact, too.

Let T be a subset of R^m. We say that T is *convex* iff for any $x, y \in T$ and $\lambda \in [0, 1]$, the convex combination $\lambda x + (1 - \lambda)y$ belongs to T. Now, let f be a function from T to T. We say that f is continuous iff for any sequence $\{x^\nu\}$ in T, if $\{x^\nu\}$ converges $x^0 \in T$, then $\{f(x^\nu)\}$ converges $f(x^0)$.

Now, we can present Brouwer's fixed point.

Theorem 1.3 (Brouwer (1908))**.** *Let T be a nonempty compact convex subset of R^m, and let f be a continuous function from T to T. Then f has a fixed point x^0 in T, i.e., $f(x^0) = x^0$.*

Proof of Theorem 1.1: Define a function $f : \prod_{i \in N} \Delta(S_i) \to \prod_{i \in N} \Delta(S_i)$ as follows:

(1) For each $s_i \in S_i$, define $g^i_{s_i}(m) = \max\{h_i(s_i, m_{-i}) - h_i(m), 0\}$.

(2) For each $s_i \in S_i$, define $f_i(m)[s_i] = \frac{m_i[s_i] + g^i_{s_i}(m)}{1 + \sum_{t_i \in S_i} g^i_{t_i}(m)}$.

It is straightforward to verify that $\sum_{s_i \in S_i} f_i(m)[s_i] = 1$ and $f_i(m)[s_i] \geq 0$ and hence $f(m) = (f_1(m), ..., f_N(m)) \in \prod_{i \in N} \Delta(S_i)$.

Now we show that if $f(m^*) = m^*$, then m^* is a Nash equilibrium. Suppose that $f(m^*) = m^*$. Fix a player $i \in N$. Because $h_i(m^*) = \sum_{s_i \in S_i} m^*_i[s_i] h_i(s_i, m^*_{-i})$, there exists some s_i^0 such that $h_i(s_i^0, m_{-i}^*) \leq h_i(m^*)$ and $m_i^*[s_i^0] > 0$. Then, $f_i(m^*) = m^*$ implies that

$$m_i^*[s_i^0] = \frac{m_i^*[s_i^0] + g_{s_i^0}^i(m^*)}{1 + \sum_{t_i \in S_i} g_{t_i}^i(m^*)} \text{ and } g_{s_i^0}^i(m^*) = 0.$$

Thus, $\sum_{t_i \in S_i} g_{t_i}^i(m^*) = 0$, and this implies that $h_i(m_i^*, m_{-i}^*) \geq h_i(m_i, m_{-i}^*)$ for any $m_i \in \Delta(S_i)$.

Finally, we show that f has a fixed point by appealing to Theorem 1.3. It is easy to verify that $\prod_{i \in N} \Delta(S_i)$ is compact and convex. It is also easy to verify that $g_{s_i}^i$ is continuous for each $s_i \in S_i$ and each $i \in N$, and hence f is continuous. $\square$

1.3 Interpretations

Steady-state interpretation

Under this interpretation, Nash equilibrium is the outcome after repeated plays. An explicit model requires a dynamic process that leads to Nash equilibrium. This literature includes learning models and evolutionary models. In this approach, the selection of Nash equilibria is based upon assumptions on the dynamic process.

Ex ante decision-making

Nash (1951) was intended to give a theory of *ex ante* decision-making from each individual player's perspective. The proposed solution is not a particular Nash equilibrium; instead, the solution is the *set* of all Nash equilibria. However, not every game is solvable.

2 Nash Noncooperative Theory

We define $E(\hat{G})$ to be the set of all Nash equilibria in $\hat{G}$. By Theorem 1.1, the set $E(\hat{G})$ is nonempty.

Let F be a subset of $\prod_{i \in N} \Delta(S_i)$. We say that F satisfies *Interchangeability* iff

$$(m_1, m_2, ..., m_N), (m_1', m_2', ..., m_N') \in F \text{ imply } (m_i', m_{-i}) \in F \text{ for all } i. \tag{3}$$

Now, let $\mathbf{E} = \{E : E \subseteq E(\hat{G})$ and E satisfies (3)$\}$. We say that E is the *Nash solution* iff E is the greatest set in $\mathbf{E}$, i.e., $E' \subseteq E$ for any $E' \in \mathbf{E}$. We say that E is a *Nash subsolution* iff E is a maximal set in $\mathbf{E}$, i.e., there is no $E' \in \mathbf{E}$ such that $E \subsetneq E'$. We call these the *Nash noncooperative solutions* from now on.

The solution exists if and only if the entire set $E(\hat{G})$ of equilibria is interchangeable. In this case, Nash called the game $\hat{G}$ *solvable*. On the other hand, a subsolution exists always; specifically, for any $(m_1, m_2, ..., m_N) \in E(\hat{G})$, there is a subsolution F with $(m_1, m_2, ..., m_N) \in F$. This is already claimed in Nash (1951).

Lemma 2.1. *For any $(m_1, m_2, ..., m_N) \in E(\hat{G})$, there is a subsolution E such that $(m_1, m_2, ..., m_N) \in E$.*

Proof. We prove for the case $N = 2$. Consider the class $\mathbf{E}(m_1, m_2) := \{E \in \mathbf{E} : E$ satisfies (3) and $(m_1, m_2) \in E\}$. This set satisfies the assumption of Zorn's lemma. Indeed, let $\{E^\lambda\}_{\lambda \in \Lambda}$ is any ascending chain in $\mathbf{E}(m_1, m_2)$. Now, let $E^o = \bigcup_{\lambda \in \Lambda} E^\lambda$. This is a set of Nash equilibria and contains (m_1, m_2). Take $(m_1', m_2'), (m_1'', m_2'')$ from E^o. Then, since $\{E^\lambda\}_{\lambda \in \Lambda}$ is an ascending chain, we have some F^λ containing $(m_1', m_2'), (m_1'', m_2'')$. Since E^λ satisfies (3), we have $(m_1', m_2'') \in E^\lambda$, *a fortiori*, $(m_1', m_2'') \in E^o$. Thus, E^o satisfies (3). We have shown that $\{E^\lambda\}_{\lambda \in \Lambda}$ has an upper bound in $\mathbf{E}(m_1, m_2)$. Hence, by Zorn's lemma, we have a maximal set E in $\mathbf{E}(m_1, m_2)$. This E is what we wanted. $\square$

For some $(m_1, m_2) \in E(\hat{G})$, a subsolution given in Lemma 2.1 may not be unique, as remarked in Jansen (1981). Consider the game given in Table 1.

Table 1

	$\mathbf{s}_{21}$	$\mathbf{s}_{22}$
$\mathbf{s}_{11}$	$(1,1)$	$(1,1)$
$\mathbf{s}_{12}$	$(1,1)$	$(0,0)$

This has two subsolutions; $\{((\alpha, 1-\alpha), (1,0)) : \alpha \in [0,1]\}$ and $\{((1,0), (\alpha, 1-\alpha)) : \alpha \in [0,1]\}$. Here, $((1,0),(1,0))$ belongs to both subsolutions.

Lemma 2.2. *Let $F \subseteq \prod_{s_i \in S_i} \Delta(S_i)$ and let $F_i = \{m_i : (m_i; m_{-i}) \in F$ for some $m_{-i} \in \Delta(S_{-i})\}$ for $i = 1, 2..., N$. Then, F satisfies (3) if and only if $F = F_1 \times F_2 \times ... \times F_N$.*

Proof. We prove for the case $N = 2$. The *if* part is straightforward. Consider the *only-if* part. First, $F \subseteq F_1 \times F_2$ holds in general. Suppose that F satisfies (3). Let $(m_1, m_2) \in F_1 \times F_2$. Then, we have some $m_1' \in \Delta(S_1)$ and $m_2' \in \Delta(S_2)$ such that $(m_1, m_2') \in F$ and $(m_1', m_2) \in F$. By (3), we have $(m_1, m_2) \in F$. □

3 Ex ante decision-making

3.1 Johansen's Postulates for the Nash Theory

Johansen (1982) gives an argument for the Nash noncooperative theory from the *ex ante* perspective. Here, we review his postulates and claims. For simplicity and without losing any insights, we consider only 2-person games.

Postulate 1. A player makes his decision $s_i \in S_i$ on the basis of, and only on the basis of information concerning the action possibility sets of two players S_1, S_2 and their payoff functions h_1, h_2.

Postulate 2. In choosing a his own decision, a player assumes that the other is rational in the same way as he himself is rational.

Postulate 3. If any decision is rational decision to make for an individual player, then this decision can be correctly predicted by the other player.

Postulate 4. Being able to predict the actions to be taken by the other player, a player's own decision maximizes his payoff function corresponding to the predicted actions of the other player.

Comments.

- Postulate 1 states that the decisions and predictions are made *before* the game starts. Here it assumes an ideal situation in which the preferences and the rules of the game are commonly known.

- Postulate 4 requires players' decisions be optimal, instead of according to a exogenous evaluation of strategies independent of other players' decisions as in the Maximin criterion, against predictions about other players' decisions.

- Postulates 2 and 3 state the principles of making predictions. In this approach, the distinction between decision and prediction is crucial.

- Rationality in Postulate 2 should be interpreted more broadly than the optimization requirement in Postulate 4; rather, it should embrace the contents of the four postulates.

3.2 Decision criterion for Nash theory

We have seen the Maximin criterion for decision-making in games. That criterion has various advantages—playability, zero-order requirements on interpersonal thinking, constructiveness, etc. However, it does not guarantee the best-response property as stated in Postulate 2 and 4; only from the outside observer's perspective, and only in zero-sum games, that criterion satisfies the best-response property to some extent.

Here we present a decision criterion based on Johansen's four postulates. Although these requirements are formulated in a symmetric manner between the two players, it should be understood as a decision criterion together with a prediction criterion for one single player.

The final decisions belong to the set $E_i \subseteq M_i$, $i = 1, 2$, that is the greatest set satisfying the following two requirements:

N1: for each $m_1 \in \text{supp}(E_1)$, $\text{Best}_1(m_1; m_2)$ holds for all $m_2 \in E_2$;

N2: for each $m_2 \in \text{supp}(E_2)$, $\text{Best}_2(m_2; m_1)$ holds for all $m_1 \in E_1$.

Comments.

- From player 1's perspective, E_1 is the *decision variable* while E_2 is the *prediction variable*. Correspondingly, condition (N1) is the decision criterion while condition (N2) is the prediction criterion.

- Although these two requirements appear as a set of simultaneous equations, it involves an infinite regress—player 1's decision in (N1) depends on his prediction about player 2's decision in (N2), which depends on her prediction about player 1's decision in (N1), *ad infinitum.*

- The choice of the quantifier before the predictions about the other player's decision reflects the assumption of *free will.*

Here we show that (N1-N2) characterizes the Nash noncooperative solution. First we give a lemma that connects (N1-N2) to Nash equilibria.

Lemma 3.1. *Let E_i be a nonempty subset of $\Delta(S_i)$ for $i = 1, 2$. Then, (E_1, E_2) satisfies N1-N2 if and only if any $(m_1, m_2) \in E_1 \times E_2$ is a Nash equilibrium in $\hat{G}$.*

Proof. (Only-If): Let (m_1, m_2) be any mixed strategy pair in $E_1 \times E_2$. By N1, any s_1 with $m_1(s_1) > 0$ gives the largest payoff over $h_1(s'_1, m_2), s'_1 \in S_1$. Hence, any mixture of those payoffs over such s_1's takes the same largest value. Hence, $h_1(m'_1, m_2)$ takes the largest value over m'_1's. By the symmetric argument, $h_2(m_1, m_2)$ takes the largest value over m'_2's. Thus, (m_1, m_2) is a Nash equilibrium in $\hat{G}$.

(If): Let $s_1 \in \sigma(E_1)$ and $m_2 \in E_2$. Then, $m_1(s_1) > 0$ for some $m_1 \in E_1$. Since (m_1, m_2) is a Nash equilibrium, we have $h_1(s_1, m_2) \geq h_1(s'_1, m_2)$ for all $s'_1 \in S_1$. Requirement N2 is shown in the parallel manner. □

The following theorem shows that the greatest set satisfies (N1-N2) corresponds to the Nash noncooperative solution if it exists.

Theorem 3.1 (The Nash Noncooperative Solutions)**.** *Let E be a subset of $\Delta(S_1) \times \Delta(S_2)$, and let $E_i = \{m_i : (m_i; m_{-i}) \in E$ for some $m_{-i} \in \Delta(S_{-i})\}$ for $i = 1, 2$. Then,*

(1): E is a Nash subsolution if and only if $E = E_1 \times E_2$ and it is a maximal set satisfying N1-N2.

(2): E is the Nash solution if and only if $E = E_1 \times E_2$ and it is the greatest set satisfying N1-N2.

Proof. (1) (If): Let $E = E_1 \times E_2$ be a maximal set satisfying N1-N2, i.e., there is no (E_1', E_2') satisfying N1-N2 with $E_1 \times E_2 \subsetneq E_1' \times E_2'$. Then E satisfies interchangeability (3). Also, $E_1 \times E_2$ is a set of Nash equilibria. Let E' be a set of Nash equilibria satisfying (3) with $E_1 \times E_2 \subseteq E'$. Then, E' is expressed as $E_1' \times E_2'$. Since $E_1' \times E_2'$ satisfies N1-N2, we have $E_1 \times E_2 = E'$ by maximality for $E_1 \times E_2$ satisfying N1-N2.

(Only-If): Since E is a subsolution, it satisfies (3). Hence, E is expressed as $E = E_1 \times E_2$. Also, $E_1 \times E_2$ satisfies N1-N2. Since $E = E_1 \times E_2$ is a subsolution, it is maximal set satisfying N1-N2.

(2) (If): Let $E = E_1 \times E_2$ be the greatest set satisfying N1-N2. E satisfies (3), and it is a set of Nash equilibria. Let E' be any set of Nash equilibria satisfying (3). Then, E' is expressed as $E_1' \times E_2'$ and satisfies N1-N2. Since $E = E_1 \times E_2$ be the greatest set satisfying N1-N2, we have $E_1' \times E_2' \subseteq E_1 \times E_2$.

(Only-If): Since E is the Nash solution, it satisfies (3). Hence, E is expressed as $E = E_1 \times E_2$. Also, since it consists of Nash equilibria, $E_1 \times E_2$ satisfies N1-N2 and is the greatest set having N1-N2. □

Comments.

- The role of interchangeability and solvability
  - interchangeability is not required in (N1-N2); rather, it is a condition on the game being played so that (N1-N2) can recommend a decision.
  - if interchangeability is not satisfied, additional requirements are necessary to derive a decision.
- Separation of players' minds
  - each player uses (N1-N2) to derive the final decision and prediction; to do so a symmetric assumption about the other player's decision and prediction criterion is necessary.
  - these assumptions lead to an infinite regress which requires common knowledge

  - these epistemic considerations can be formalized in *epistemic logic*

- The role of mixed strategies

  - mixed strategies are objects of choice in our formulation and warrantees the existence of a *subsolution*

  - conceptually, however, the use of mixed strategies is not essential; rather, it creates some difficulties