# Game Theory: Lecture 2

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Outline

- Two-person zero-sum games
  - normal-form games
  - Minimax theorem
  - Simplex method

# 1 2-person 0-sum games

## 1.1 2-Person Normal Form Games

A 2-person *normal form game* is given as a triple:

$$G = (N, \{S_i\}_{i \in N}, \{h_i\}_{i \in N}),$$

where

(1): $N = \{1, 2\}-$ the set of players;

(2): $S_i = \{\mathbf{s}_{i1}, ..., \mathbf{s}_{i\ell_i}\}$ $-$ the set of pure strategies for player $i = 1, 2$;

(3): $h_i : S_1 \times S_2 \to \mathbb{R}$ $-$ the payoff function of player $i = 1, 2$.

A 2-person normal form game $G = (N, \{S_i\}_{i\in N}, \{h_i\}_{i\in N})$ is often described by a matrix form:

Prisoner's Dilemma

| | $\mathbf{s}_{21}$ | $\mathbf{s}_{22}$ |
|---|---|---|
| $\mathbf{s}_{11}$ | $(5,5)$ | $(1,6)$ |
| $\mathbf{s}_{12}$ | $(6,1)$ | $(3,3)$ |

Matching Pennies

| | $\mathbf{s}_{21}$ | $\mathbf{s}_{22}$ |
|---|---|---|
| $\mathbf{s}_{11}$ | $(1,-1)$ | $(-1,1)$ |
| $\mathbf{s}_{12}$ | $(-1,1)$ | $(1,-1)$ |

We say that a 2-person game is *zero-sum* iff

$$h_1(s_1, s_2) + h_2(s_1, s_2) = 0 \text{ for all } (s_1, s_2) \in S_1 \times S_2. \tag{1}$$

**Observation.** Suppose that $h_1(s_1, s_2) + h_2(s_1, s_2) = 0$ for all $(s_1, s_2) \in S_1 \times S_2$ and suppose that $h_1$ and $h_2$ represent the preference relation $\succsim_1$ and $\succsim_2$ on $\Delta(S_1 \times S_2)$. Then for any $p, q \in \Delta(S_1 \times S_2)$, $p \succsim_1 q$ if and only if $q \succsim_2 p$.

## 1.2 Maximin Decision Criterion

We consider the individual decision making criterion for player $i$ called the *maximin decision criterion.* It has two steps:

(1): Player $i$ evaluates each of his strategies by its worst possible payoff;

(2): Player $i$ maximizes the evaluation by controlling his strategies.

These two steps are formulated mathematically as follows. Let $i = 1$.

(1*): for each $s_1 \in S_1$, the evaluation of $s_1$ is defined by $\min_{s_2} h_1(s_1, s_2)$;

(2*): Player 1 maximizes $\min_{s_2} h_1(s_1, s_2)$ by controlling $s_1$.

These two steps are expressed by

$$\max_{s_1\in S_1} \min_{s_2\in S_2} h_1(s_1, s_2) = \max_{s_1\in S_1}(\min_{s_2\in S_2} h_1(s_1, s_2)). \tag{2}$$

We say that $s_1^*$ is a *maximin strategy* iff it is a solution of (2).

**Example 1**: Consider the following zero-sum game:

| | $\mathbf{s}_{21}$ | $\mathbf{s}_{22}$ | $\min_{s_2} h_1(s_1, s_2)$ |
|---|---|---|---|
| $\mathbf{s}_{11}$ | $(5, -5)$ | $(4, -4)$ | 4 |
| $\mathbf{s}_{12}$ | $(3, -3)$ | $(6, -6)$ | 3 |
| $\min_{s_1} h_2(s_1, s_2)$ | ? | ? | |

When the zero-sum condition (1) holds, the maximization of $h_1$ is equivalent to the minimization of $h_2$, i.e.,

$$h_1(s_1, s_2) \to \max_{s_1} \quad \Longleftrightarrow \quad h_2(s_1, s_2) \to \min_{s_1} \tag{3}$$

and the minimization of $h_1$ is equivalent to the maximization of $h_2$, i.e.,

$$h_1(s_1, s_2) \to \min_{s_2} \quad \Longleftrightarrow \quad h_2(s_1, s_2) \to \max_{s_2}. \tag{4}$$

By (3) and (4), the maximin decision criterion for player 2 will be formulated as follows:

($1^*$-2): for each $s_2 \in S_2$, the evaluation of $s_2$ is defined by $\max_{s_1} h_1(s_1, s_2)$;

($2^*$-2): Player 2 minimizes $\max_{s_1} h_1(s_1, s_2)$ by controlling $s_2$.

These two steps are expressed by

$$\min_{s_2 \in S_2} \max_{s_1 \in S_1} h_1(s_1, s_2) = \min_{s_2 \in S_2} (\max_{s_1 \in S_1} h_1(s_1, s_2)). \tag{5}$$

**Lemma 1.1.** $\max_{s_1 \in S_1} \min_{s_2 \in S_2} h_1(s_1, s_2) \leq \min_{s_2 \in S_2} \max_{s_1 \in S_1} h_1(s_1, s_2)$.

*Proof.* . Let $t_1 \in S_1$ and $t_2 \in S_2$ be arbitrary strategies for players 1 and 2. First, we have

$$h_1(t_1, t_2) \leq \max_{s_1 \in S_1} h_1(s_1, t_2)$$

Looking at the latter inequality, we have

$$\min_{s_2 \in S_2} h_1(t_1, s_2) \leq \min_{s_2 \in S_2} \max_{s_1 \in S_1} h_1(s_1, s_2).$$

Since the right-hand side is constant, we have

$$\max_{s_1 \in S_1} \min_{s_2 \in S_2} h_1(s_1, s_2) \leq \min_{s_2 \in S_2} \max_{s_1 \in S_1} h_1(s_1, s_2).$$

$\square$

In the following example, the assertion of Lemma 1.1 holds in inequality.

*Example* 1.1. : Consider the zero-sum game:

| | $\mathbf{s}_{21}$ | $\mathbf{s}_{22}$ | $\min_{s_2} h_1(s_1, s_2)$ |
|---|---|---|---|
| $\mathbf{s}_{11}$ | $5\ (-5)$ | $3\ (-3)$ | 3 |
| $\mathbf{s}_{12}$ | $2\ (-2)$ | $6\ (-6)$ | 2 |
| | | | $\max_{s_1} \min_{s_2} h_1(s_1, s_2) = 3$ |
| $\max_{s_1} h_1(s_1, s_2)$ | 5 | 6 | $\min_{s_2} \max_{s_1} h_1(s_1, s_2) = 5$ |

In the following example, the assertion of Lemma 1.1 holds in equality.

*Example* 1.2. : Consider the zero-sum game:

| | $\mathbf{s}_{21}$ | $\mathbf{s}_{22}$ | $\min_{s_2} h_1(s_1, s_2)$ |
|---|---|---|---|
| $\mathbf{s}_{11}$ | 5 | 3 | 3 |
| $\mathbf{s}_{12}$ | 6 | 4 | 4 |
| | | | $\max_{s_1} \min_{s_2} h_1(s_1, s_2) = 4$ |
| $\max_{s_1} h_1(s_1, s_2)$ | 6 | 4 | $\min_{s_2} \max_{s_1} h_1(s_1, s_2) = 4$ |

*Example* 1.3. : The Scissors-Rock-Paper is formulated as follows:

| | **Sc** | **Ro** | **Pa** |
|---|---|---|---|
| **Sc** | 0 | $-1$ | 1 |
| **Ro** | 1 | 0 | $-1$ |
| **Pa** | $-1$ | 1 | 0 |

Calculate the maximin value and minimax value.

## 1.3 Strictly Determined Games

**Definition 1.1.** We say that a 2-person zero-sum game $G = (N, \{S_i\}_{i\in N}, \{h_i\}_{i\in N})$ is *strictly determined* iff

$$\max_{s_1\in S_1}\min_{s_2\in S_2} h_1(s_1, s_2) = \min_{s_2\in S_2}\max_{s_1\in S_1} h_1(s_1, s_2). \tag{6}$$

In fact, the equation (6) is related to some concept of an equilibrium. In a 2-person zero-sum game, it is the saddle point, but in general, it is the Nash equilibrium.

**Definition 1.2.** Let $G = (N, \{S_i\}_{i\in N}, \{h_i\}_{i\in N})$ be a 2-person normal form game. We say that $(s_1^*, s_2^*)$ is a *saddle point* with respect to $h_1$ iff for all $s_1 \in S_1$ and $s_2 \in S_2$

$$h_1(s_1, s_2^*) \leq h_1(s_1^*, s_2^*) \leq h_1(s_1^*, s_2). \tag{7}$$

In fact, this coincidence of having a saddle point and (6) is not mere accidental, as it can be easily verified in previous examples. We have the following theorem.

**Theorem 1.1.** *(1): A 2-person game $G$ has a saddle point if and only if (6) holds.*

*(2): Suppose that $G$ has a saddle point. Then a pair $(s_1^*, s_2^*)$ of strategies is a saddle point if and only if $s_1^*$ and $s_2^*$ are maximin strategies for players* 1 *and* 2, *respectively.*

*Proof.* We prove the Only-If part and If part of (1). These are also proofs of the Only-If part and If part of (2).

(1)(*Only-If*): Let $(s_1^*, s_2^*)$ be a saddle point of game $G$. Then, $h_1(s_1^*, s_2^*) = \max_{s_1} h_1(s_1, s_2^*) = \min_{s_2} h_1(s_1^*, s_2)$ by (7). Consider $\min_{s_2}\max_{s_1} h_1(s_1, s_2^*)$. Then, $\min_{s_2}\max_{s_1} h_1(s_1, s_2^*) \leq h_1(s_1^*, s_2^*)$. Similarly, we have $\max_{s_1}\min_{s_2} h_1(s_1, s_2^*) \geq h_1(s_1^*, s_2^*)$. By Lemma 1.1, we have (6).

(*If*): Suppose that (6) holds. Let $s_1^*$ and $s_2^*$ be maximin strategies for players 1 and 2. By (6), we have

$$\max_{s_1} h_1(s_1, s_2^*) = \min_{s_2}\max_{s_1} h_1(s_1, s_2) = \max_{s_1}\min_{s_2} h_1(s_1, s_2) = \min_{s_2} h_1(s_1^*, s_2)$$

If $h_1(s_1^*, s_2^*) \leq \max_{s_1} h_1(s_1, s_2^*)$ and $\min_{s_2} h_1(s_1^*, s_2) \leq h_1(s_1^*, s_2^*)$, we have $\max_{s_1} h_1(s_1, s_2^*) \leq h_1(s_1^*, s_2^*) \leq \min_{s_2} h_1(s_1^*, s_2)$. This implies (7). □

The concept of a saddle point is, in fact, equivalent to the concept of a Nash equilibrium. We say that a strategy pair $(s_1^*, s_2^*)$ is a *Nash equilibrium* iff

$$h_1(s_1, s_2^*) \leq h_1(s_1^*, s_2^*) \text{for all } s_1 \in S_1, \tag{8}$$

$$h_2(s_1^*, s_2) \leq h_2(s_1^*, s_2^*) \text{for all } s_2 \in S_{2.}$$

We have the following theorem on the relationship between the saddle point and Nash equilibrium.

**Theorem 1.2.** *Let $G = (N, \{S_i\}_{i \in N}, \{h_i\}_{i \in N})$ be a zero-sum two-person game. A pair $(s_1^*, s_2^*)$ of strategies is a saddle point for $h_1$ if and only if $(s_1^*, s_2^*)$ is a Nash equilibrium in $G$.*

*Proof.* The first half of (8) is the same as the first half of (7). Hence, it suffices to compare the latter half of (8) with that of (7). By the zero-sum condition (1), the latter half of (8) is rewritten as

$$-h_1(s_1^*, s_2) \leq -h_1(s_1^*, s_2^*) \text{for all } s_2 \in S_{2.}$$

This is equivalent to

$$h_1(s_1^*, s_2^*) \leq h_1(s_1^*, s_2) \text{ for all } s_2 \in S_{2.}$$

This is the latter half of (7). □

As we have seen, a saddle point may not exist. von Neumann (1928) introduces *mixed strategies* and extend a finite normal form game to include those strategies. Formally, for a given normal-form game $G = \langle N, \{S_i\}_{i \in N}, \{h_i\}_{i \in N} \rangle$, the mixed extension, denoted by $\hat{G}$, is the triple $\langle N, \{M_i\}_{i \in N}, \{h_i\}_{i \in N} \rangle$ such that $M_i = \Delta(S_i)$ for each $i = 1, 2$ and $h_i$ is the von Neumann-Morgenstern expected utility indices over $\Delta(S_1 \times S_2)$. For 0-sum games, it turns out that the game $\hat{G}$ always has a saddle point.

**Theorem 1.3.** *Let $\hat{G}$ be the mixed extension of a 2-person 0-sum game $G$. Then,*

$$\max_{m_1 \in M_1} \min_{m_2 \in M_2} h_1(m_1, m_2) = \min_{m_2 \in M_2} \max_{m_1 \in M_1} h_1(m_1, m_2). \tag{9}$$

It is easy to verify that both Theorem 1.1 and Theorem 1.2 hold for the game $\hat{G}$. Hence, Theorem 1.3 is equivalent to the existence of a saddle point.

**Theorem 1.4.** *Let $E(\hat{G})$ be the set of saddle points of the mixed extension $\hat{G}$, and let $E_i(\hat{G}) = \mathrm{Proj}_{S_i} E(\hat{G})$ for $i = 1, 2$. Then $E(\hat{G}) = E_1(\hat{G}) \times E_2(\hat{G})$ and each $E_i(\hat{G})$ is a convex set.*

*Proof.* We show that if $m_1^*$ solves

$$\max_{m_1 \in M_1} \min_{m_2 \in M_2} h_1(m_1, m_2)$$

and $m_2^*$ solves

$$\min_{m_2 \in M_2} \max_{m_1 \in M_1} h_1(m_1, m_2),$$

then $(m_1^*, m_2^*)$ is a saddle point. By (9), we have

$$\max_{m_1} h_1(m_1, m_2^*) = \min_{m_2} \max_{m_1} h_1(m_1, m_2) = \max_{m_1} \min_{m_2} h_1(m_1, m_2) = \min_{m_2} h_1(m_1^*, m_2)$$

Let the common value be $v$. Then,

$$\text{for any } n_1 \in M_1, h_1(n_1, m_2^*) \leq \max_{m_1 \in M_1} h_1(m_1, m_2^*) = v$$

and

$$\text{for any } n_2 \in M_2, h_1(m_1^*, n_2) \geq \min_{m_2} h_1(m_1^*, m_2) = v.$$

But $h_1(m_1^*, m_2^*) \geq \min_{m_2 \in M_2} h_1(m_1^*, m_2) = v$ and $h_1(m_1^*, m_2^*) \leq \max_{m_1 \in M_1} h_1(m_1, m_2^*) = v$.

On the other hand, let $(m_1^*, m_2^*)$ be a saddle point of game $\hat{G}$. Then, $h_1(m_1^*, m_2^*) = \max_{m_1} h_1(m_1, m_2^*) = \min_{m_2} h_1(m_1^*, m_2)$ by (7). Consider $\min_{m_2} \max_{m_1} h_1(m_1, m_2)$. Then, $\min_{m_2} \max_{m_1} h_1(m_1, m_2) \leq h_1(m_1^*, m_2^*)$. Similarly, we have $\max_{m_1} \min_{m_2} h_1(m_1, m_2) \geq h_1(m_1^*, m_2^*)$. By Lemma 1.1, we have (9). Moreover, for any $m_1$,

$$\min_{m_2 \in M_2} h_1(m_1, m_2) \leq h_1(m_1, m_2^*) \leq h_1(m_1^*, m_2^*) = \min_{m_2 \in M_2} h_1(m_1^*, m_2)$$

and for any $m_2$,

$$\max_{m_1 \in M_1} h_1(m_1, m_2) \geq h_1(m_1^*, m_2) \geq h_1(m_1^*, m_2^*) = \max_{m_1 \in M_1} h_1(m_1, m_2^*).$$

Finally we show that $E_1(\hat{G})$ is convex. Suppose that $m_1, m_1' \in E_1(\hat{G})$. Then,

$$\min_{m_2 \in M_2} h_1(m_1, m_2) = \min_{m_2 \in M_2} h_1(m_1', m_2).$$

Thus, for any $m_2 \in M_2$,

$$\alpha h_1(m_1, m_2) + (1-\alpha) h_1(m_1', m_2) \geq \alpha \min_{m_2 \in M_2} h_1(m_1, m_2) + (1-\alpha) \min_{m_2 \in M_2} h_1(m_1', m_2),$$

and hence $\alpha m_1 + (1-\alpha) m_1'$ is also a maximin strategy. □

## 1.4 Maximin Criterion as a Solution

### 1.4.1 Conceptual evaluations

*A priori* demand on a theory of rational behavior in games:

1. Coherent decision criterion: optimal against predictions about opponent's behavior.

The Maximin criterion is coherent in the sense that from individual's perspective, inability to make perfect predictions implies maximization of security levels.

2. Perfect rationality: predictability of opponent's behavior.

Although the Maximin criterion does not require predictability, it is compatible with it for 0-sum games.

3. Interpersonal knowledge and intrapersonal logical ability.

The Maximin criterion only requires self-consciousness.

4. Playability: implementation of the decision criterion.

The solution to the Maximin criterion is constructive through linear programming.

**Epistemic extensions.**

The Maximin criterion no interpersonal knowledge requirements. However, in zero-sum games, the Minimax theorem shows that if both players employ the Maximin criterion (denoted by $MMi$), then the resulted decisions satisfies the decision criteria (N1) and (N2) in Hu-Kaneko. It is then a curious fact whether these two decision criteria are equivalent under the assumption of common knowledge of the game being zero-sum. Formally, the desirable conclusions include

$$C(g), C(MM1 \wedge MM2) \vdash (I_1(s_1) \wedge I_2(s_2)) \equiv C(Nash(s_1, s_2))$$

and

$$C(g) \vdash C(MM1 \wedge MM2) \equiv C(D1 \wedge D2).$$

First of all, we need to supplement $MMi$ with $D01i - D02i$. Then, the first statement above seems a restatement of the Minimax theorem in epistemic terms; the second statement seems more involved and perhaps is not provable. One related issue is weather common knowledge is necessary for the above derivations, and it seems to be the case. If it actually requires common knowledge, then the *epistemic* status of Maximin decision criterion as a way to achieve the best-response property ($D3i$) is not far from the Nash noncooperative solution. The only difference lies more or less on the playability aspect—the Maximin criterion can be solved by the simplex method. Nonetheless, the whole exercise is to give a separate argument (than Aumman's consideration of the Maximin criterion and criticism of it) to bridge the difference between the Maximin criterion and the Nash noncooperative solution.

### 1.4.2 Maximin Criterion and Linear Programming

An important feature of 0-sum games is that the solution to the Maximin criterion can be constructively obtained. This is done by transforming the maximization problem in the Maximin criterion into a linear programming problem.

Let $\hat{G} = \langle N, \{M_i\}_{i \in N}, \{h_i\}_{i \in N} \rangle$ be the mixed extension of a 2-person 0-sum game. Assume, without loss of generality, that $h_1(s_1, s_2) > 0$ for all $(s_1, s_2) \in S_1 \times S_2$. Consider the following problem:

$$\min_{u_{s_1}, s_1 \in S_1} \sum_{s_1 \in S_1} u_{s_1}$$

subject to

$$u_{s_1} \geq 0 \text{ for all } s_1 \in S_1, \ \sum_{s_1 \in S_1} u_{s_1} h_1(s_1, s_2) \geq 1 \text{ for all } s_2 \in S_2.$$

**Lemma 1.2.** *(1) The above problem is feasible, i.e., there exists $\{u_{s_1} : s_1 \in S_1\}$ that satisfies the constraints.*

*(2) If $\{u^*_{s_1} : s_1 \in S_1\}$ solves the above problem, then $m_1 \in M_1$ defined as*

$$m_1^*[s_1] = \frac{u^*_{s_1}}{\sum_{s_1 \in S_1} u^*_{s_1}}$$

*solves the Maximin criterion.*

### 1.4.3 Linear Programming

First we prove that linear programming is a finite problem, that is, either the problem is unbounded or it is sufficient to check extreme points for optimality. We consider a linear programming in the following form: **Problem 1.**

$$\min_{x \in \mathbb{R}^n} c^T x = \sum_{j=1}^{n} c_j x_j \text{ subject to } Ax = b, \ x \geq 0,$$

where $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$ is a matrix with rank $m$, and $b \in \mathbb{R}^m$.

**Definition 1.3.** A *hyperplane* in $\mathbb{R}^n$ is a set of the form

$$H = \{x \in \mathbb{R}^n : a_1 x_1 + .... + a_n x_n = b_0\},$$

with $a_i \neq 0$ for some $i \in \{1, ..., n\}$.
A *half-space* is a set of the form

$$\{x \in \mathbb{R}^n : a_1 x_1 + .... + a_n x_n \geq b_0\},$$

with $a_i > 0$ for some $i$.
A *polyhedron* is the intersection of finitely many half-spaces.

We denote the $j$th column of the matrix $A$ by $A_j$. Let $\{A_{j_1}, ..., A_{j_m}\}$ be a set of $m$ linearly independent columns in $A$. Let $B = [A_{j_1}, ..., A_{j_m}]$, and let $y = B^{-1}b$. The vector $x \in \mathbb{R}^n$ defined by $x_{j_r} = y_r$ for $r = 1, ..., m$ and $x_j = 0$ if $j \notin \{j_1, ..., j_m\}$ is called a *basic solution* w.r.t. the basis $\{j_1, ..., j_m\}$ to Problem 1. A basic solution $x$ is called a *basic feasible solution*, abbreviated as BSF, if $x \geq 0$. Let

$$F = \{x \geq 0 : Ax = b\}$$

be the set of feasible solutions to Problem 1.

**Definition 1.4.** A point $w$ in $F = \{x \geq 0 : Ax = b\}$ is called a *vertex* if for any $x \neq w$ and $y \neq w$ and any $\lambda \in [0, 1]$, $w \neq \lambda x + (1 - \lambda)y$.

**Theorem 1.5.** *A vector $w \in \mathbb{R}^n$ is a vertex of $F$ if and only if it is a basic feasible solution of Problem 1.*

*Proof.* ($\Rightarrow$) Let $w$ be a vertex of $F$ and let $I = \{i : 1 \leq i \leq n, w_i > 0\}$. First we show that the set $\{A_i : i \in I\}$ is linearly independent, and then we show that $w$ is a BFS.

(1) $\{A_i : i \in I\}$ is linearly independent. Suppose not and suppose that $\sum_{i \in I} \lambda_i A_i = 0$ for some $\{\lambda_i\}_{i \in I}$, $\lambda_{i_0} \neq 0$ for some $i_0 \in I$. Define $\lambda_j = 0$ for $j \notin I$. Then $A\lambda = 0$ and $\lambda_j \neq 0$ only if $w_j > 0$.

Let $\theta = \min_{\lambda_j \neq 0} \frac{w_j}{|\lambda_j|} > 0$. Let $w^+ = w + \theta\lambda$ and $w^- = w - \theta\lambda$. Then, $Aw^+ = A(w + \theta\lambda) = Aw + \theta(A\lambda) = b$; similarly, $Aw^- = b$. Now we show that $w^+ \geq 0$ and $w^- \geq 0$. For each $j \notin I$, $w_j^+ = w_j \geq 0$. For each $i \in I$, if $\lambda_i = 0$ then $w_i^+ = w_i \geq 0$; if $\lambda_i \neq 0$ then

$$w_i^+ = w_i + \theta\lambda_i \geq w_i - \lambda_i \frac{w_i}{\lambda_i} = 0;$$

similarly, $w_i^- \geq 0$.

Thus, $w^+ \in F$ and $w^- \in F$. Because $\lambda_{i_0} \neq 0$, $w^+ \neq w$ and $w^- \neq 0$. But $w = \frac{1}{2}w^+ + \frac{1}{2}w^-$, a contradiction to $w$ being a vertex of $F$.

(2) $w$ is a BFS if $\{A_i : i \in I\}$ is linearly independent. Because $\{A_i : i \in I\}$ is linearly independent, $|I| \leq m$. Expand $I$ into $J$ if necessary so that $I \subseteq J$, $|J| = m$, $\{A_j : j \in J\}$

is linearly independent. Because $Aw = b$, $w$ is a basic solution w.r.t. $\{A_j : j \in J\}$. $w$ is feasible because $w \in F$.

($\Leftarrow$) Suppose that $w$ is a BFS w.r.t. $\{A_j : j \in J\}$. Suppose, to the contrary, that $w = \lambda u + (1-\lambda)v$ for some $u \neq w$, $v \neq w$, $\lambda \in (0,1)$, and $u, v \in F$.

Because $w$ is a BFS, then for any $w' \in F$ such that $w'_j = 0$ for all $j \notin J$, $w' = w$. Now, because $u \in F$, $u_j = 0$ for all $j \notin J$. So $u = w$, a contradiction. $\square$

**Definition 1.5.** A BFS $w$ for problem 1 is *degenerate* if $|\{i : w_i = 0\}| > n - m$.

**Theorem 1.6.** *If two different bases correspond to a single BFS $w$, then $w$ is degenerate.*

*Proof.* Let $I$ and $J$ be the two bases. Then $|I \cap J| < m$ and $w_i = 0$ for all $i \notin I \cap J$. Hence, $w$ is degenerate. $\square$

**Lemma 1.3.** *Suppose that $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ is a polyhedron. Then $v$ is a vertex of $P$ if and only if there are $n$ linearly independent constraints among $Ax \leq b$ that are tight at $w$.*

*Proof.* ($\Rightarrow$) Suppose that $v$ is a vertex of $P$. Let $A \in \mathbb{R}^{m \times n}$, and we denote the $i$th row of $A$ by $A^i$. Suppose, to the contrary, that the set $I = \{i : A^i v = b_i\}$ has rank $r < n$. Then the space $H = \{t \in \mathbb{R}^n : A^i t = 0 \text{ for all } i \in I\}$ has dimension $n - r$. Let $t \neq 0$ be a nonzero vector in $H$. Then, for sufficiently small $\theta > 0$, $A^j(v \pm \theta t) < b_j$ for all $j \notin I$ and $A^i(v \pm \theta t) = b_i$ for all $i \in I$. Thus, $v \pm \theta t \in P$ and $v \pm \theta t \neq v$ because $\theta > 0$ and $t \neq 0$. But $v = \frac{1}{2}(v + \theta t) + \frac{1}{2}(v - \theta t)$, a contradiction to $v$ being a vertex of $P$.

($\Leftarrow$) Let $\{A^{i_1}, ..., A^{i_n}\}$ be the set of linearly independent constraints that are tight at $v$. Let $B$ be a $n \times n$ matrix with $B^r = A^{i_r}$ for $r = 1, ..., n$. Let $b' \in \mathbb{R}^n$ be the vector such that $b'_r = b_{i_r}$ for $r = 1, ..., n$. Then $v = B^{-1}b'$.

Suppose that $v = \lambda u + (1-\lambda)w$ for some $u, w \in P$ and $\lambda \in [0,1]$. Then $Bu \leq b'$ and $Bw \leq b'$. But $b' = Bv = \lambda Bu + (1-\lambda)Bw$, and hence $Bu = b' = Bw$. Thus, $u = B^{-1}b' = v = w$. $\square$

Now we show that either the Problem 1 has an optimal solution among the vertexes or it is unbounded.

**Theorem 1.7.** *Suppose that* $v \in F = \{x \in \mathbb{R}^n : Ax = b, \; x \geq 0\}$*. Then, either for any* $M \in \mathbb{R}$*, there exists* $y \in F$ *such that* $c \cdot y < M$*, or there exists a vertex* $w$ *of* $F$ *such that* $c \cdot w \leq c \cdot v$*.*

*Proof.* $v \in F$ implies $Av = b$ and $v \geq 0$. Let $(C_j)$ be the constraint $x_i \geq 0$ for $j = 1, ..., n$ and let $(E_i)$ be the constraint $A^i x = b_i$ for $i = 1, ..., m$. Let $S \subseteq \{C_1, ..., C_n, E_1, ..., E_m\}$ be the largest subset of constraints including $\{E_1, ..., E_m\}$ that are linearly independent and are tight at $v$. Let $|S| = r$. If $r = n$, then by Lemma 1.3 $v$ is a vertex and we are done. Therefore assume that $r < n$. Let $I = S \cap \{C_1, ..., C_n\}$.

Notice that if $v_j = 0$ and $j \notin S$, then the constraint $(C_j)$ can be written as a linear combination of constraints in $S$. Now let

$$F^* = \{x \in \mathbb{R}^n : x_j = 0 \text{ for all } j \in I, \; Ax = 0\}.$$

$dim(F^*) = (n - m) - (r - m) = n - r.$

$\square$