# Contracting with Heterogeneous Externalities

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# Rewards and Incentives

- In a team environment rewards may affect performance in a non-monotonic way.
- The argument is not based on any behavioral or psychological assumption.
  - Gneezy Rustichini (2000)
  - Falk and Fehr (2001)
- The argument builds on the externalities among peers.

# Introduction

- What is a multi agent initiative?
  - An activity relies on agents' participation
  - Externalities arise among agents
  - The principal offers a set of optimal incentives for participation

# Throwing a Party

- There is a group of participants whom you would like all to come to the party
- $w_i(j)$ is how much person $i$ enjoys the fact that person $j$ attends the party

- How should you lure the group effectively taking into account the externalities?

# Acquisition

- A firms makes acquisition offers to several owners of other firms.
- A decision of an owner whether to sell or not depends on whom of his rivals is expected to be purchased.
- How should the acquiring firm go about making these offers?

# Network Technologies

- A buyer's (agent's) willingness to pay for the technology is affected not only by the number of buyers that are expected to adopt the technology but also by the identity of these buyers
- How should the producer introduce the product into the market

# The Mall

- A mall owner tries to lease stores
- Big department stores or brand name store attract costumers more than other store and thus induce positive externalities on these stores.

- How should the mall owner price leases?
  - We will bring some real evidence later

# An Interesting Empirical Result

- Gould et al (2005)
- Anchor stores (such as department stores, stores with national brand name, etc) generate large positive externalities by attracting most of the customer traffic to the mall, and therefore increase the sales of non-anchor stores.
- On average, anchor stores occupy over 58% of the total leasable space in the mall and yet pay only 10% of the total rent collected by the mall's owner.

# Introduction

- Main Assumptions:
  - Heterogeneous Agents
  - Externalities arise between agents, when the externality induced on agent 1 due to the participation of agent 2 is the utility benefit/loss of agent 1 due mutual participation with agent 2.
  - The principal maximizes profit
  - Two distinguished stages: Selection and Participation
  - The principal assumes worst case scenario

# Main Questions

- What is the optimal mechanism that sustains full participation?
- What determines the reward of each agent?
- How specific attributes of the externalities among agents affect the principal's costs?
- How does the solution of the participation problem would affect agents' interrelations prior to the selection stage?

# Surprising Connection

- How do sport associations transform tournaments outcomes into teams' ranking (the NCAA for collegiate football)
- Condorcet proposal of how to rank candidates based on election outcome.

# The Model

- A participation problem $(N, w, c)$
  - $N$ - Set of Agents
  - $w$ - Externalities Structure where $w_i(j) \in \mathbf{R}$: $\begin{bmatrix} 0 & w_1(2) & w_1(3) \\ w_2(1) & 0 & w_2(3) \\ w_3(1) & w_3(2) & 0 \end{bmatrix}$
  - $c$ - Outside options vector
- Solution: incentive mechanism $v = (v_1, \ldots, v_n)$
- Additively separable preferences of the agents
- Agent $i$ participates iff $\sum_{j \in C} w_i(j) + v_i \geq c$

# The Model

- Agents' decisions are taken simultaneously
- Incentive mechanism $v$ is incentive inducing (INI) if full participation is a unique Nash equilibrium in the normal form game $G(v)$.
- The optimal mechanism (the solution) is the INI mechanism with the minimal sum of rewards.

# Positive Externalities

- The set of "Divide and Conquer" (DAC) mechanisms have the following structure:

$$v = \left(c, c - w_{i_2}(i_1), c - w_{i_3}(i_1) - w_{i_3}(i_2), \cdots, c - \sum_k w_{i_n}(i_k)\right)$$

when $\varphi = \{i_1,\ i_2,\ \ldots,\ i_n\}$ is an arbitrary ranking.

Proposition 1: If $v$ is an optimal mechanism, then it is a DAC mechanism

# Positive Externalities – Optimal Ranking

Definition: Tournament $G(N, A)$ is a simple and complete directed graph, where $N$ is the set of vertices and $A$ is the set of arcs.

- All matches between the agents are described by a tournament $G(N, A)$ where $N$ is the set of agents and the definition of $A$ is:

$$w_i(j) < w_j(i) \Leftrightarrow (i, j) \in A$$

$$w_i(j) = w_j(i) \Leftrightarrow (i, j) \in A \text{ and } (j, i) \in A$$

# Optimal Ranking of Acyclic Tournaments

Definition: A ranking is consistent if every pair of agents satisfy that if agent 1 beats agent 2 then agent 1 precedes 2 in the ranking.

Lemma 1: If tournament $G(N,A)$ is acyclic then there exists a unique ranking which is consistent with tournament $G(N,A)$ (henceforth the tournament ranking)

Proposition 2: Let $(N,\ w,\ c)$ be a participation problem for which the corresponding tournament $G(N,A)$ is acyclic. Let $\varphi$ be the tournament ranking, then the optimal mechanism is given by the DAC mechanism with respect to ranking $\varphi$

# The Intuition Behind Proposition 2:

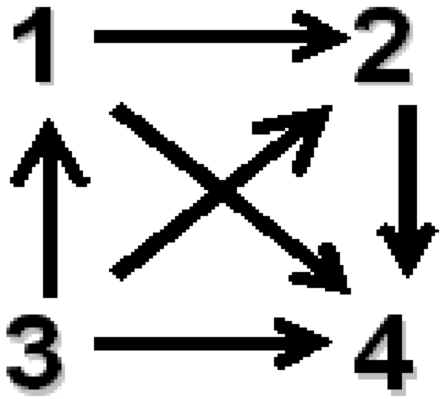
$$v = \left(c, c - w_{i_2}(i_1), c - w_{i_3}(i_1) - w_{i_3}(i_2), \cdots, c - \sum_k w_{i_n}(i_k)\right)$$

- Assume agent 1 beats agent 2, i.e., agent 1 likes agent 2 more than 2 likes 1 hence $w_1(2) < w_2(1)$
  - If agent 1 precedes 2, agent 2's reward is reduced by $w_2(1)$
  - If agent 2 precedes 1, agent 1's reward is reduced by $w_1(2)$
- Therefore, agent 1 should precede 2 to minimize the principal's payments.

# Example

- $N = \{1, 2, 3, 4\}$
- $c = 20$

$$w = \begin{bmatrix} 0 & 1 & 3 & 2 \\ 4 & 0 & 2 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 5 & 2 & 0 \end{bmatrix}$$




- The optimal ranking $\varphi = (3, 1, 2, 4)$
- The optimal DAC mechanism $v = (20, 17, 14, 10)$

# Externalities Structure's Characteristics under Acyclic

- Aggregate Externalities Level: $K_{agg} = \frac{1}{2}\sum_{i,j} w_i(j)$
- Asymmetry Level: $K_{asym} = \frac{1}{2}\sum_{i<j} |w_i(j) - w_j(i)|$

Proposition 3: Let $(N, w, c)$ be a participation problem and $V$ the sum of incentives of the optimal mechanism $v$. If the corresponding tournament $G(N,A)$ is acyclic then: $V = n \cdot c - K_{agg} - K_{asym}$

# Corollaries

Corollary 1: When the externalities structure is symmetric i.e., $w_i(j) = w_j(i)$, then all DAC mechanisms are optimal.

Corollary 2: Let $V$ be the minimal payment of a participation problem $(N, w, c)$, then $V$ is strictly decreasing with the asymmetry level.

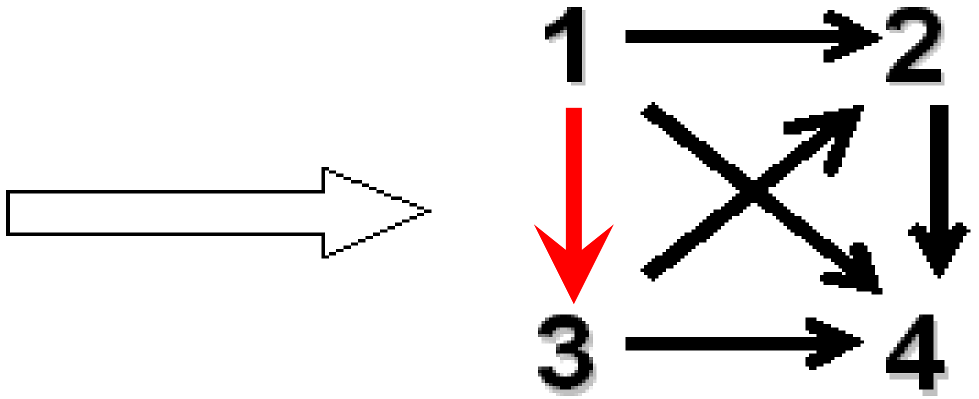
# Implication

- $K_{asym}$ is a measure of the degree of mutuality in the relationships between the agents. The principal benefits from a low level of mutuality as it allows him to utilize the externalities among the agents

## Example

- $N = \{1, 2, 3, 4\}$
- $c = 20$

$$w = \begin{bmatrix} 0 & 1 & 3 & 2 \\ 4 & 0 & 2 & 2 \\ 3.1 & 1 & 0 & 1 \\ 3 & 5 & 2 & 0 \end{bmatrix}$$




- The optimal ranking $\varphi = (1, 3, 2, 4)$
- The optimal DAC mechanism $v = (20, 16.9, 14, 10)$

# What Else?

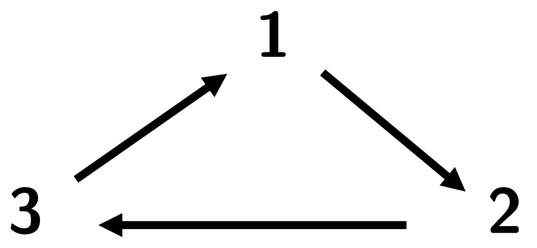
- Solution for cyclic tournaments and the inconsistency problem.
- Solution for Negative Externalities participation problem (full compensation)
- Solution for the Mixed Externalities problem (two stage solution)

# Optimal Ranking in Cyclic Tournaments

- Sport Associations
- Condorcet a la Peyton Young

# Optimal Ranking of Cyclic Tournaments

- If Tournament $G(N,A)$ is cyclic then no consistent ranking exists.




- Definition 1: An inconsistency arise from pair of agents 1 and 2 If agent 1 precedes agent 2, although agent 2 beats 1.
  - Various Sports organizations (such as the NCAA for collegiate football) provide ranking on cyclic tournament.
  - Solution procedures offered by Operation Research literature is MVR ("minimal violations ranking")

# Optimal Ranking of Cyclic Tournaments

- Consider two agents 1 and 2 that satisfy an inconsistency and assume that agent 1 beats agent 2 $(w_1(2) < w_2(1))$
  - If agent 1 precedes agent 2 → payments are reduced by $w_2(1)$
  - If agent 2 precedes agent 1 → payments are reduced by $w_1(2)$

- Therefore, inconsistency cost arise and equal to:

$$t_1(2) = w_2(1) - w_1(2) \text{ when } (1,2) \in A$$

# Optimal Ranking of Cyclic Tournaments

- Definition 3: For each subset of arcs $S$ we define

$$t(S) = \sum_{(i,j)\in S} t(i,j)$$

- Definition 4: Let $G_{-S}$ the graph obtained from $G$ by reversing the arcs in subset $S \subset A$.
- Definition 5: Consider a cyclic graph $G$ and let $S^*$ be the subset of arcs that satisfy:
  1. $G_{-S^*}$ is acyclic
  2. $t(S^*) \leq t(S)$ for each $S$ that satisfy condition 1.

# Optimal Ranking of Cyclic Tournaments

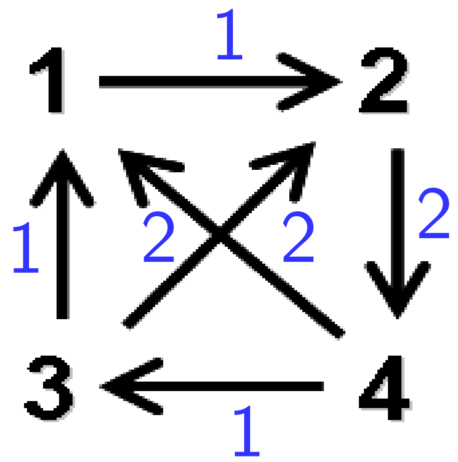
**Proposition 4:** Let $(N, w, c)$ be a participation problem with a cyclic tournament $G$. Let $\varphi$ be the tournament ranking of $G_{-S^*}$. Then, the optimal mechanism is the DAC mechanism with respect to $\varphi$

**Proposition 5:** Let $(N, w, c)$ be a participation problem and $V$ the sum of incentives of the optimal mechanism $v$. If the corresponding tournament $G(N,A)$ is cyclic then: $V = n \cdot c - K_{agg} - K_{asym} + t(S^*)$

# Example

- $N = \{1, 2, 3, 4\}$
- $c = 20$

$$w = \begin{bmatrix} 0 & 1 & 2 & 5 \\ 2 & 0 & 4 & 2 \\ 1 & 2 & 0 & 7 \\ 3 & 4 & 6 & 0 \end{bmatrix}$$




- $S_1{}^* = \{(2, 4)\} \rightarrow \varphi = (4, 3, 1, 2) \rightarrow v = (20, 13, 13, 12)$
- $S_2{}^* = \{(1, 2), (3, 4)\} \rightarrow \varphi = (3, 2, 4, 1) \rightarrow v = (20, 16, 10, 12)$

# Negative Externalities

- Negative externalities arise mainly in "congestion markets" such as Competition among applicant, Market entry, etc.

Proposition 6: Let $(N, w, c)$ be a participation problem with negative externalities, then the unique optimal mechanism $v$ is:

$$v_i = c - \sum_{i \neq j} |w_i(j)|$$

# Mixed Externalities (Two Stage Solution)

Proposition 8: Let $v$ be the optimal mechanism of participation problem $(N, w, c)$.

Consider a positive participation problem $(N, q, c)$ such that $q_i(j) = w_i(j)$ if $w_i(j) > 0$ and $q_i(j) = 0$ if $w_i(j) < 0$.

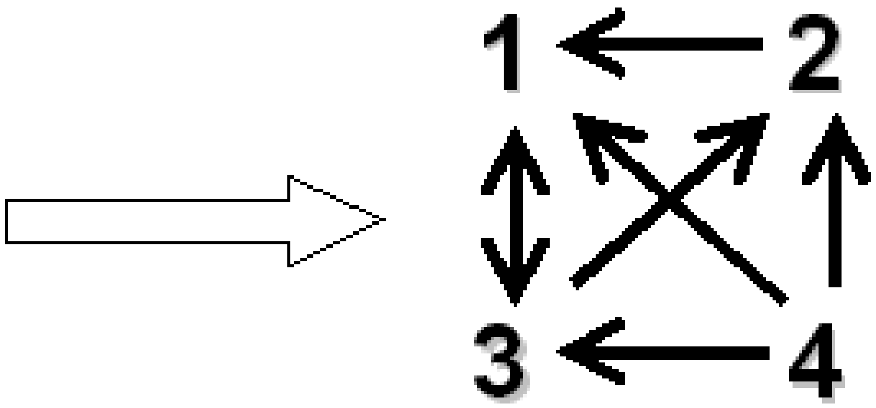
Then,

$$v_i = u_i - \sum_{j \in D_i} |w_i(j)| \text{ where } D_i = \{j | w_i(j) < 0 \text{ s.t. } i, j \in N\}$$

## Example

- $N = \{1, 2, 3, 4\}$
- $c = 20$

$$w = \begin{bmatrix} 0 & 2 & 0 & 3 \\ 0 & 0 & 3 & 9 \\ 0 & 1 & 0 & 4 \\ 1 & 2 & 3 & 0 \end{bmatrix}$$




- Optimal ranking of the positive problem: $\varphi = (4, 3, 2, 1)$
- Optimal mechanism of the positive problem: $u = (20, 16, 3, 15)$
- Note that: $S^* = \{(1, 3)\}$
  - Optimal mechanism: $v = (20, 16{+}4, 3{+}1, 15{+}2) = (20, 20, 4, 17)$

## Some Examples

- Segregation - agents benefit from participating with their own group's members and enjoy no benefit from participating with members from the other group.
  - More specifically, consider the two groups $B_1$ and $B_2$ such that for each $i, j \in B_k$ , $k = 1, 2$, we have $w_i(j) = 1$.
  - Otherwise, $w_i(j) = 0$.

## Some Examples

- Desegregation - agents benefit from participating with the other group's members and enjoy no benefit from participating with members of their own group.
  - More specifically, consider the two groups $B_1$ and $B_2$ such that for each $i, j \in B_k$ , $k = 1, 2$, we have $w_i(j) = 0$.
  - Otherwise, $w_i(j) = 1$.

## Some Examples

- Status - the society is partitioned into two status groups, high and low. Each member of the society benefits from participating with each member of the high status group and enjoys no benefits with members of the low status group.
  - Formally, let $B_1$ be the high status group and set $w_i(j) = 1$ if $j \in B_1$ and otherwise $w_i(j) = 0$.

# Proposition 9

- Let $(N, w, c)$ be a participation problem. Let $n_1$ and $n_2$ be the number of agents selected from groups $B_1$ and $B_2$ respectively such that $n_1+n_2 = n$.
- Denote by $v(n_1,n_2)$ the principal cost of incentivizing agents under the optimal mechanism given that the group composition is $n_1$ and $n_2$. The following holds:
  1. under Segregation $v(n_1,n_2)$ is declining in $|n_1 - n_2|$.
  2. under Desegregation $v(n_1,n_2)$ is increasing in $|n_1 - n_2|$.
  3. under Status $v(n_1,n_2)$ is increasing in $n_1$.

# Extensions

- Sequential Participation Decisions
- Non-Additive Externalities
- Variable Outside Options

Thank you