# **Epistemic Logic and Applications**

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T-W Hu(Bristol U)	lecture 3	December 14, 2019 1/28
	Propositional logic	

## Logical system

Logical inferences are crucial in game theoretical arguments

- derivation of best responses
- derivation of others' best responses and then equilibrium

Symmetry is an important assumption in social sciences

- the analyst assumes the subjects are symmetric to himself in many ways
- including the logical abilities

#### Logical inferences

Mathematical logic treats logical inferences as objects of study

- an inference is simply a sequence of symbols
- but follows a certain rules

Connection between provability and validity

- a statement is provable if there is a proof for it
- a statement is valid if it is true in all states of the world

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	Propositional logic	

# Propositional logic

Simplest setting to study logical inferences

- begins with a set of *elementary* or atomic propositions
- each statement consists of elementary statements connected by logical connectives

Belief operators to distinguish different players' scopes of thinking

- discuss logical inferences within each player's scope
- the analyst makes inference about the objective world

## Propositions

A set of elementary propositions,  $\mathcal{P}_0$ 

- typical element denoted by p, q, r
- interpreted as "indecomposable" propositions

A set of logical connectives

- ∨, or
- $\wedge$ , and
- $\neg$ , negation
- $\Rightarrow$ , implication

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	Propositional logic		

# Propositions (cont.)

The set of all (well-formulated) propositions is defined by induction

- $\bullet \ \ \mathsf{base} \ \ \mathsf{is} \ \mathcal{P}_0$
- induction step: construct new propositions from previous layers by connection by logical connectives

Formally, the set  ${\mathcal P}$  is generated by finite applications of:

- if  $A \in \mathbf{P}_0$ , then  $p \in \mathcal{P}$
- if  $A, B \in \mathcal{P}$ , then  $A \lor B, A \land B, \neg A, A \Rightarrow B \in \mathcal{P}$

Example:  $p \Rightarrow (q \Rightarrow p)$  is constructed from

- first,  $A = (q \Rightarrow p)$  from p and q
- then,  $p \Rightarrow A$

### Syntax vs Semantics

Syntax is concerned with the "form" of a proposition

- $p \wedge q$  and  $q \wedge p$  are syntactically different
- but they seem to have the same meaning

Semantics is concerned with the "meaning" of a proposition

• formalized by truth assignment

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	Completeness		

## Truth assignment

A truth assignment is a function  $\tau : \mathcal{P}_0 \to \{\top, \bot\}$ 

•  $\top$  means "true,  $\perp$  means "false"

 $\tau$  can then be extended to  $\mathcal P$  by induction

- if  $\tau(A) = \top$ , or  $\tau(B) = \top$ , then  $\tau(A \lor B) = \top$ ; o/w,  $\tau(A \lor B) = \bot$
- if  $\tau(A) = \top = \tau(B)$ , then  $\tau(A \land B) = \top$ ; o/w,  $\tau(A \land B) = \bot$
- if  $\tau(A) = \top$ , then  $\tau(\neg A) = \bot$ ;  $\mathsf{o}/\mathsf{w}$ ,  $\tau(A \lor B) = \top$
- if  $\tau(A) = \bot$ , or  $\tau(B) = \top$ , then  $\tau(A \Rightarrow B) = \top$ ; o/w,  $\tau(A \Rightarrow B) = \bot$

#### Validity

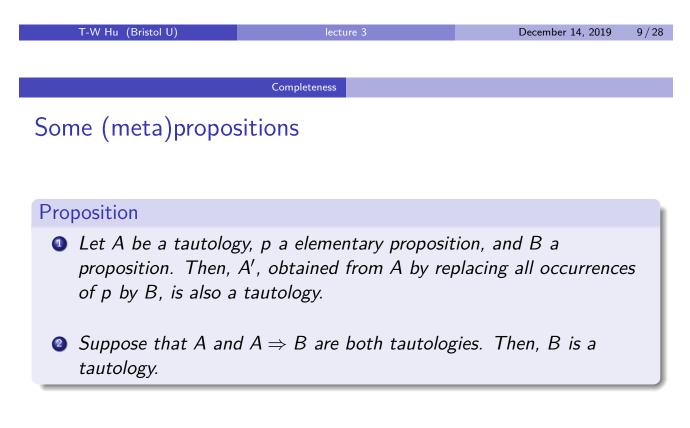
A proposition A is a *tautology* if

 $\tau(A) = \top$  under any truth assignment  $\tau$ 

Examples:

- $p \Rightarrow (q \Rightarrow p)$
- $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$
- $((\neg p) \Rightarrow (\neg q)) \Rightarrow (((\neg p) \Rightarrow q) \Rightarrow p)$

If  $A \in \mathcal{P}$  contains *n* elementary propositions, how many verifications do you need to check validity of *A*?



• these are *meta*-propositions, propositions about propositional logic, *not* propositions within propositional logic

#### Examples

Let  $A, B, C \in \mathcal{P}$ ; show that the followings are tautologies:

- $A \Rightarrow (B \Rightarrow A)$
- $(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$
- $((\neg A) \Rightarrow (\neg B)) \Rightarrow (((\neg A) \Rightarrow B) \Rightarrow A)$

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	Completeness	

# Proof Theory

Formally, a proof is a sequence of propositions

- each item is either an axiom
- or follows from previous items a according to a *inference rule*

#### Axioms: let $A, B, C \in \mathcal{P}$

- L1  $A \Rightarrow (B \Rightarrow A)$
- L2  $(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$
- L3  $((\neg A) \Rightarrow (\neg B)) \Rightarrow (((\neg A) \Rightarrow B) \Rightarrow A)$

Inference rule: from  $A \Rightarrow B$  and A infer B (MP, modus ponens)

#### Completeness

#### Proof

A sequence of propositions,  $\{A_1, A_2, ..., A_n\}$  is a *proof* of B if

- $A_n = B$
- for each i = 1, ..., n, either
  - $A_i$  is an axiom, or
  - $A_i$  is obtained from  $A_{i'}$  and  $A_{i''}$  using MP, i', i'' < i
- We use  $\vdash B$  to denote the fact that B is provable

Let  $\Gamma \subset \mathcal{P}$ ; we use  $\Gamma \vdash B$  to denote the fact that

- there is proof for *B*, in which
- propositions in Γ can be used as axioms

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	Completeness		

### Examples

- For any  $A \in \mathcal{P}$ , the proposition  $A \Rightarrow A$  is provable
- S1 let  $B = (A \Rightarrow A)$ ; then, L2 implies  $(A \Rightarrow (B \Rightarrow A)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow A))$
- S2 but L1 implies  $A \Rightarrow ((A \Rightarrow A) \Rightarrow A)$ , that is,  $A \Rightarrow (B \Rightarrow A)$
- S3 then, from S1 and S2, MP implies  $(A \Rightarrow B) \Rightarrow (A \Rightarrow A)$
- S4 L1 implies  $A \Rightarrow (A \Rightarrow A)$ , that is,  $A \Rightarrow B$
- S5 from S3 and S4, MP implies  $A \Rightarrow A$

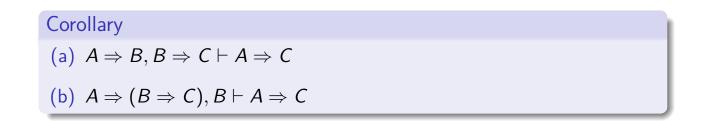
#### Completeness

### Deduction Theorem

Theorem

Let  $\Gamma \subset \mathcal{P}$  and  $A, B \in \mathcal{P}$ . If  $\Gamma, A \vdash B$ , then  $\Gamma \vdash A \Rightarrow B$ .

• as a corollary,  $A \vdash B$  if and only if  $\vdash A \Rightarrow B$ 



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	Completeness		
Soundness Theoren	n		

Theorem If B is provable, then B is a tautology.

#### Completeness

## **Completeness Theorem**

#### Theorem

If B is a tautology, then B is provable.

The proof uses the following lemma

#### Lemma

Let  $A \in \mathcal{P}$  and let  $B_1, ..., B_n$  be the elementary propositions that occur in A. For any truth assignment  $\tau$ , define

$$B'_i = B_i$$
 if  $\tau(B_i) = \top$ , and  $B'_i = \neg B_i$  if  $\tau(B_i) = \bot$ .

Similarly, define A' = A if  $\tau(A) = \top$  and  $A' = \neg A$  otherwise. Then,

$$B'_1, \ldots, B'_n \vdash A'.$$

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	Completeness		

Consistency

The propositional logic is *consistent* in the sense that for any  $B \in \mathcal{P}$ , it cannot be the case that

$$\vdash B \text{ and } \vdash \neg B$$

A set of propositions,  $\Gamma$ , is *consistent* if, for any *B*, it is not the case that

$$\Gamma \vdash B$$
 and  $\Gamma \vdash \neg B$ 

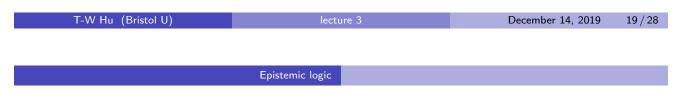
### Epistemic logic

A framework for formal epistemology

- different thinking scopes for different individuals
- describing different individuals' beliefs
- describe different individuals' inferences

Potential applications to game theory and economics

- formalize the notion "common knowledge" or the lack of it
- formalize bounded interpersonal reasoning, e.g., level-k theory



### Belief operators

Set of individuals (players): i = 1, ..., N

- each player is capable of logical inferences
- we use  $\mathbf{B}_i$ , the belief operator, to describe the scope of *i*'s thinking

Set of propositions,  $\mathcal{L}$ :

- if A and B are propositions, so are  $A \Rightarrow B$ ,  $A \Rightarrow B$ ,  $\neg A$ , and  $A \land B$
- if A is a proposition, so is  $\mathbf{B}_i(A)$

For example,  $\mathbf{B}_1(\mathbf{B}_2(A))$  is a proposition

• it describes player 1's belief about player 2's belief

This is a finite language, but includes higher order beliefs of arbitrary orders

## Epistemic proof theory

Axioms for propositional logic, (L1)-(L3), and MP

• all tautologies are provable

Epistemic axioms: for all i = 1, ..., N

$$\mathsf{K} \; \mathbf{B}_i(A \Rightarrow B) \Rightarrow (\mathbf{B}_i(A) \Rightarrow \mathbf{B}_i(B))$$

$$\mathsf{D} \neg \mathsf{B}_i(A \land \neg A)$$

Epistemic inference rule: Nec from A infer  $\mathbf{B}_i(A)$ 

- axiom K and rule Nec ensure that the player has perfect logical ability
- axiom D ensures that player *i*'s beliefs are consistent

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Epistemic logic				
Epistemic logic				
		Epistemic logic		

## Kripke semantics

Extends the truth assignment  $\tau$  to belief operators

- to do so, need to have a scope for each player
- Kripke semantics uses connection between different possible worlds to model the scopes

A Kripke model is a list  $M = (W, P_1, ..., P_N, \tau)$ 

- set of possible worlds, W (set of states)
- accessibility relation for each i,  $P_i$  (possibility relation)
- truth valuation:  $\tau: W \times \mathcal{P}_0 \rightarrow \{\top, \bot\}$

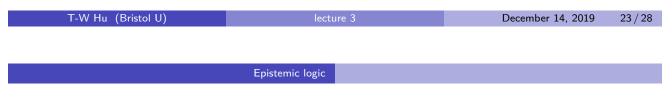
#### Epistemic logic

#### Truth evaluation

The function  $\tau$  is extended to  $W \times \mathcal{L}$  as follows:

- $\tau(w, A) = \top$  iff  $\tau(w, \neg A) = \bot$
- $\tau(w, A \land B) = \top$  iff  $\tau(w, A) = \top = \tau(w, B)$
- $\tau(w, A \Rightarrow B) = \top$  iff  $\tau(w, A) = \bot$  or  $\tau(w, B) = \top$
- $\tau(w, \mathbf{B}_i(A)) = \top$  iff  $\tau(v, A) = \top$  for all v such that  $(w, v) \in P_i$

A is valid (denoted  $\models_M A$ ) under M iff  $\tau(w, A) = \top$  for all w



### Some meta-theorems

A proposition A is non-epistemic if it does not contain any belief operator

Theorem

- (a) Let A be a non-epistemic proposition. Then, A is valid under any M if and only if A is a tautology.
- (b) Let  $A \in \mathcal{L}$  and let M be a model. If A is valid under M, so is  $\mathbf{B}_i(A)$ .
- (c) Let  $A, B \in \mathcal{L}$  and let M be a model. Then,  $\mathbf{B}_i(A \Rightarrow B) \Rightarrow (\mathbf{B}_i(A) \Rightarrow \mathbf{B}_i(B))$  is valid under M.

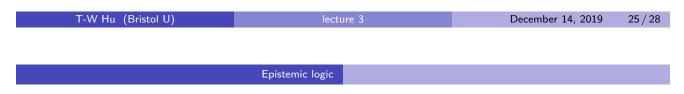
Construct a model M such that  $\mathbf{B}_i(p \wedge \neg p)$  is valid under M

#### Completeness theorem

A model *M* is *serial* if for all *i* and for all  $w \in W$ , there exists *v* such that  $(w, v) \in P_i$ 

Theorem

For any proposition  $A \in \mathcal{L}$ ,  $\vdash A$  if and only if  $\models_M A$  for any M that is serial.



Other epistemic axioms

- $\top \mathbf{B}_i(A) \Rightarrow A$
- 4  $\mathbf{B}_i(A) \Rightarrow \mathbf{B}_i(\mathbf{B}_i(A))$
- 5  $\neg \mathbf{B}_i(A) \Rightarrow \mathbf{B}_i(\neg \mathbf{B}_i(A))$

Interpretations

- axiom T connects playeri's belief to the outer world
- axioms 4 and 5 impose introspection, positive and negative

In the literature

- $\bullet$  system with K+T+4 is called S4 system
- system with K+T+5 is called S5 system, or *partition model*

#### Completeness theorem for S4 and S5

A model M is called

- transitive if for all i and u, v, w ∈ W, (u, v) ∈ P<sub>i</sub> and (v, w) ∈ P<sub>i</sub> imply (u, w) ∈ P<sub>i</sub>
- reflexive if for all i and  $w \in W$ ,  $(w, w) \in P_i$
- euclidean if for all i and  $u, v, w \in W$ ,  $(u, v) \in P_i$  and  $(u, w) \in P_i$ imply  $(v, w) \in P_i$

Note that  $P_i$  is an equivalence relation if and only if it is transitive, reflexive, and euclidean

