Normal-from games

Tai-Wei Hu

University of Bristol

2-person normal-form game

A 2-person *normal form game* is given as a triple:

$$G = (N, \{S_i\}_{i \in N}, \{h_i\}_{i \in N}),$$

where

(1): $N = \{1, 2\}-$ the set of players;

(2): $S_i = \{\mathbf{s}_{i1}, ..., \mathbf{s}_{i\ell_i}\}$ $-$ the set of pure strategies for player $i = 1, 2$;

(3): $h_i : S_1 \times S_2 \to \mathbb{R}$ $-$ the payoff function of player $i = 1, 2$.

Matrix form

A 2-person normal form game $G = (N, \{S_i\}_{i \in N}, \{h_i\}_{i \in N})$ is often described by a matrix form:

Prisoner's Dilemma

	$\mathbf{s}_{21}$	$\mathbf{s}_{22}$
$\mathbf{s}_{11}$	$(5, 5)$	$(1, 6)$
$\mathbf{s}_{12}$	$(6, 1)$	$(3, 3)$

Matching Pennies

	$\mathbf{s}_{21}$	$\mathbf{s}_{22}$
$\mathbf{s}_{11}$	$(1, -1)$	$(-1, 1)$
$\mathbf{s}_{12}$	$(-1, 1)$	$(1, -1)$

Zero-sum game

We say that a 2-person game is *zero-sum* iff

$$h_1(s_1, s_2) + h_2(s_1, s_2) = 0 \text{ for all } (s_1, s_2) \in S_1 \times S_2. \quad (1)$$

- in a zero-sum game, if h_1 and h_2 represent the preference relation $\succsim_1$ and $\succsim_2$ on $\Delta(S_1 \times S_2)$, for any $p, q \in \Delta(S_1 \times S_2)$,

$$p \succsim_1 q \Leftrightarrow q \succsim_2 p$$

Maximin decision criterion

Two-step evaluation:
(1): Player i evaluates each of his strategies by its worst possible payoff

(2): Player i maximizes the evaluation by controlling his strategies

Mathematically: for $i = 1$,

(1*): for each $s_1 \in S_1$, the evaluation of s_1 is defined by $\min_{s_2} h_1(s_1, s_2)$;

(2*): Player 1 maximizes $\min_{s_2} h_1(s_1, s_2)$ by controlling s_1.

These two steps are expressed by

$$\max_{s_1 \in S_1} \min_{s_2 \in S_2} h_1(s_1, s_2) = \max_{s_1 \in S_1}\left(\min_{s_2 \in S_2} h_1(s_1, s_2)\right). \tag{2}$$

We say that s_1^* is a *maximin strategy* iff it is a solution of (2).

Example 1

Consider the following zero-sum game:

	$\mathbf{s}_{21}$	$\mathbf{s}_{22}$	$\min_{s_2} h_1(s_1, s_2)$
$\mathbf{s}_{11}$	$(5, -5)$	$(4, -4)$	4
$\mathbf{s}_{12}$	$(3, -3)$	$(6, -6)$	3
$\min_{s_1} h_2(s_1, s_2)$	?	?	

Maximization of h_1 is equivalent to minimization of h_2, i.e.,

$$h_1(s_1, s_2) \to \max_{s_1} \iff h_2(s_1, s_2) \to \min_{s_1} \tag{3}$$

and minimization of h_1 is equivalent to maximization of h_2, i.e.,

$$h_1(s_1, s_2) \to \min_{s_2} \iff h_2(s_1, s_2) \to \max_{s_2}. \tag{4}$$

Maximin Criterion (cont.)

By (3) and (4), the maximin decision criterion for player 2 is then:

(1*-2): for each $s_2 \in S_2$, the evaluation of s_2 is defined by $\max_{s_1} h_1(s_1, s_2)$;

(2*-2): Player 2 minimizes $\max_{s_1} h_1(s_1, s_2)$ by controlling s_2

Mathematically,

$$\min_{s_2 \in S_2} \max_{s_1 \in S_1} h_1(s_1, s_2) = \min_{s_2 \in S_2} (\max_{s_1 \in S_1} h_1(s_1, s_2)). \qquad (5)$$

Lemma

$\max_{s_1 \in S_1} \min_{s_2 \in S_2} h_1(s_1, s_2) \leq \min_{s_2 \in S_2} \max_{s_1 \in S_1} h_1(s_1, s_2)$.

In the following example, the assertion of Lemma 1 holds in inequality.

Example

Consider the zero-sum game

	$\mathbf{s}_{21}$	$\mathbf{s}_{22}$	$\min_{s_2} h_1(s_1, s_2)$
$\mathbf{s}_{11}$	5 (−5)	3 (−3)	3
$\mathbf{s}_{12}$	2 (−2)	6 (−6)	2
			$\max_{s_1} \min_{s_2} h_1(s_1, s_2) = 3$
$\max_{s_1} h_1(s_1, s_2)$	5	6	$\min_{s_2} \max_{s_1} h_1(s_1, s_2) = 5$

In the following example, the assertion of Lemma 1 holds in equality.

Example

Consider the zero-sum game

	$\mathbf{s}_{21}$	$\mathbf{s}_{22}$	$\min_{s_2} h_1(s_1, s_2)$
$\mathbf{s}_{11}$	5	3	3
$\mathbf{s}_{12}$	6	4	4
			$\max_{s_1} \min_{s_2} h_1(s_1, s_2) = 4$
$\max_{s_1} h_1(s_1, s_2)$	6	4	$\min_{s_2} \max_{s_1} h_1(s_1, s_2) = 4$

Example

The Scissors-Rock-Paper game

	Sc	**Ro**	**Pa**
Sc	0	-1	1
Ro	1	0	-1
Pa	-1	1	0

Calculate the maximin value and minimax value.

Strictly Determined Games

Definition

A 2-person zero-sum game $G = (N, \{S_i\}_{i \in N}, \{h_i\}_{i \in N})$ is *strictly determined* iff

$$\max_{s_1 \in S_1} \min_{s_2 \in S_2} h_1(s_1, s_2) = \min_{s_2 \in S_2} \max_{s_1 \in S_1} h_1(s_1, s_2). \tag{6}$$

Mixed strategies

As seen above, not all zero-sum games have equilibrium

- mathematically, the issue is lack of convexity

von Neumann (1928) introduced *mixed strategies*

- the mixed extension of G is to replace S_i by $M_i = \Delta(S_i)$
- h_i is the von Neumann-Morgenstern expected utility indices over $\Delta(S_1 \times S_2)$

Three interpretations of mixed strategies

- as implemented with randomized devices
- as beliefs over other's strategies
- as unpredictable strategies

Equivalence

Theorem

The following statements are equivalent.

1. *The game G has an equilibrium point.*
2. $\max_{m_1 \in M_1} \min_{m_2 \in M_2} h_1(m_1, m_2) = \min_{m_2 \in M_2} \max_{m_1 \in M_1} h_1(m_1, m_2)$.
3. *There exist $m_1^* \in M_1$ and $m_2^* \in M_2$ and $v \in \mathbb{R}$ such that*

$$h_1(m_1^*, s_2) \geq v \text{ for all } s_2 \in S_2; \tag{7}$$

$$h_1(s_1, m_2^*) \leq v \text{ for all } s_1 \in S_1. \tag{8}$$

The Minimax Theorem

Theorem

Let $\hat{G}$ be the mixed extension of a 2-person 0-sum game G. Then,

$$\max_{m_1 \in M_1} \min_{m_2 \in M_2} h_1(m_1, m_2) = \min_{m_2 \in M_2} \max_{m_1 \in M_1} h_1(m_1, m_2). \tag{9}$$

Proof using linear programming

Assume that $h_1(s_1, s_2) > 0$ for all (s_1, s_2); consider the following problem:

$$\min_{\{u_{s_1} : s_1 \in S_1\}} \sum_{s_1 \in S_1} u_{s_1} \tag{10}$$

$$\text{s.t.} \quad u_{s_1} \geq 0 \text{ for all } s_1 \in S_1, \sum_{s_1 \in S_1} u_{s_1} h_1(s_1, s_2) \geq 1 \text{ for all } s_2 \in S_2 \tag{11}$$

Lemma

(1) There exists $\{u_{s_1} : s_1 \in S_1\}$ *that satisfies (10)*

(2) If $\{u^*_{s_1} : s_1 \in S_1\}$ *solves (10)-(11), then* $m_1 \in M_1$ *defined as*

$$m^*_1(s_1) = \frac{u^*_{s_1}}{\sum_{s_1 \in S_1} u^*_{s_1}}$$

solves the Maximin criterion.

N-Person Normal Form Games

A N-person *normal form game* is given as a triple:

$$G = (N, \{S_i\}_{i \in N}, \{h_i\}_{i \in N}),$$

where
(1): $N = \{1, 2, ..., N\}$—the set of players;

(2): $S_i = \{\mathbf{s}_{i1}, ..., \mathbf{s}_{i\ell_i}\}$—the set of pure strategies for player $i = 1, 2, ..., N$;

(3): $h_i : S_1 \times S_2 \to \mathbb{R}$—the payoff function of player $i = 1, 2, ..., N$.

The following is the famous theorem due to John F. Nash.

Theorem (Nash (1951))

Let $G = (N, \{S_i\}_{i \in N}, \{h_i\}_{i \in N})$ be a N-person finite normal form game. Then, the mixed extension $\hat{G} = (N, \{\Delta(S_i)\}_{i \in N}, \{h_i\}_{i \in N})$ has a Nash equilibrium.

Theorem 9 is proved by applying Brouwer's fixed point theorem (or Kakutani's fixed point theorem)

Euclidean space

R^m, m-dimensional Euclidean space, has metric d

$$d(x, y) = \sqrt{\sum_{t=1}^{m} (x_t - y_t)^2} \text{ for } x, y \in R^m$$

A sequence $\{x^\nu\}$ *converges to* x^0, denoted by $x^\nu \to x^0$, if the sequence $\{d(x^\nu, x^0)\}$ converges to 0

Compactness

Two topological notions:

- $T \subseteq R^m$ is *closed* if for any sequence $\{x^\nu\}$ in T, $\{x^\nu\} \to x^0$ implies that $x^0 \in T$
- $T \subseteq R^m$ is *bounded* if there is a number M such that $d(0, x) \leq M$ for all $x \in T$

$T \in R^m$ is *compact* iff T is closed and bounded

- the interval $[0, 1]$ is compact
- the m-dimensional simplex is compact

Convexity and continuity

$T \subset \mathbb{R}^m$ is *convex* if for any $x, y \in T$ and $\lambda \in [0, 1]$, the convex combination $\lambda x + (1 - \lambda)y \in T$

A function $f : T \to T$ is continuous if for any sequence $\{x^\nu\}$ in T, $x^\nu \to x^0$, then $f(x^\nu) \to f(x^0)$

Brouwer's fixed point theorem

Theorem (Brouwer (1908))

Let T be a nonempty compact convex subset of R^m, and let f be a continuous function from T to T. Then f has a fixed point x^0 in T, i.e., $f(x^0) = x^0$.

Interpretations

Steady-state interpretation

Ex ante decision-making

Prediction and undecidability

Prediction/decision making in game theory

Payoff interdependence

- one player's optimal choice depends on other players' actions
- prediction about others' actions crucial to one's decision

Battle of Sexes

	Board Game	*Hiking*
Board Game	(3, 2)	(0, 0)
Hiking	(0, 0)	(2, 3)

How to make predictions?

Give up making predictions

- dominant strategy criterion, default choice

Prediction by induction from past experiences

- treating players as nature and use probability distributions
- evolutionary game theory/learning theory

Prediction by inferences

- infer others' actions from their preferences and decision methods
- *ex ante* prediction-making is a process of logical inferences

Formal theory of inferences: proof theory

Proof theory treats "proofs" as mathematical objects

- a proof is a sequence of symbols, each element is either an *axiom*, or is derived from preceding elements following a *rule*
- a sentence A is provable, denoted by $\vdash A$, if a proof for A exists

Proof theory connected to model theory by completeness theorem

- completeness: for all sentences A,

$$\vdash A \text{ if and only if } A \text{ is "true" in every model}$$

Our proof theory approach highlights an undecidability result for prediction/decision making in games, using model theory as a tool

Logical inferences and interpersonal beliefs

Logical inferences in game situations

- *ex ante* considerations require subjective inference for each player
- one player's inference may require simulated inferences for others

Epistemic logic: proof-theoretical approach to prediction-making in games

- *belief operators* to model a player's subjective scope
- *epistemic axioms* to model simulated inferences

Players make decisions and predictions based on beliefs about preferences and decision criterion

Prediction/decision criterion

Decision criterion based on payoff maximization w.r.t. predictions

- possible final decision if best response against predicted actions
- independent decision-making: take *all* predictions into account

Nash theory

- symmetric prediction/decision criterion
- prediction based on inference from other's decision criterion
- requires an infinite regress of beliefs

Can a player reach a final decision from this infinite regress?

Undecidability in prediction/decision making

Let Γ_i represent player i's beliefs (or infinite regress) of preferences and decision criteria and let $\mathbf{I}_1(s_1)$ mean "s_1 is a possible final decision"

- Γ_i leads to decidability if for each s_i,
  - $\mathbf{B}_i(\Gamma_i) \vdash \mathbf{B}_i(\mathbf{I}_i(s_i))$ (positive decision), or
  - $\mathbf{B}_i(\Gamma_i) \vdash \mathbf{B}_i(\neg\mathbf{I}_i(s_i))$ (negative decision)
- Γ_i leads to undecidability if for some s_i,
  - $\mathbf{B}_i(\Gamma_i) \nvdash \mathbf{B}_i(\mathbf{I}_i(s_i))$ and $\mathbf{B}_i(\Gamma_i) \nvdash \mathbf{B}_i(\neg\mathbf{I}_i(s_i))$

We characterize

- the class of games for which Nash theory leads to decidability
- the class of games for which Nash theory leads to undecidability

Example: decidable case

	L	R_1	R_2
U	$(5, 5)$	$(1, 0)$	$(1, 0)$
D_1	$(0, 1)$	$(2, -2)$	$(-2, 2)$
D_2	$(0, 1)$	$(-2, 2)$	$(2, -2)$

Under Nash theory,

- $\mathbf{B}_1(\Gamma_1) \vdash \mathbf{B}_1(\mathbf{I}_1(U))$
- $\mathbf{B}_1(\Gamma_1) \vdash \mathbf{B}_1(\neg\mathbf{I}_1(D_1)) \wedge \mathbf{B}_1(\neg\mathbf{I}_1(D_2))$

Example: undecidable case

	L	R
U	(3, 2)	(0, 0)
D	(0, 0)	(2, 3)

Under Nash theory,

- $\mathbf{B}_1(\Gamma_1) \nvdash \mathbf{B}_1(\mathbf{I}_1(U))$, $\mathbf{B}_1(\Gamma_1) \nvdash \mathbf{B}_1(\neg\mathbf{I}_1(U))$
- $\mathbf{B}_1(\Gamma_1) \nvdash \mathbf{B}_1(\mathbf{I}_1(D))$, $\mathbf{B}_1(\Gamma_1) \nvdash \mathbf{B}_1(\neg\mathbf{I}_1(D))$

Nash Theory

Nash solution of noncooperative games

$G = \langle \{1,2\}, \{S_1, S_2\}, \{h_1, h_2\} \rangle$, a two-person finite game

- $E \subseteq S_1 \times S_2$ is interchangeable iff $E = E_1 \times E_2 \neq \emptyset$
- interchangeability captures independence of players' decision-making
- E_i describes player i's decisions and E_j describes his predictions

Solvable and unsolvable games (Nash, 1951)

- G is solvable if $E(G)$ (the set of Nash equilibria) is interchangeable and $E(G)$ is the solution
- otherwise, G is unsolvable
  - maximal $E \subseteq E(G)$ satisfying interchangeability is a *subsolution*

Decision criterion for Nash solutions

A candidate solution $E = E_1 \times E_2 \subset S$ satisfies

N$_1$ If $s_1 \in E_1$, then s_1 is a best response against all $s_2 \in E_2$;

N$_2$ If $s_2 \in E_2$, then s_2 is a best response against all $s_1 \in E_1$.

- for player 1, E_1 describes his "good" decisions and E_2 his predictions
- N$_2$ and N$_2$ can be viewed as a system of simultaneous equations

Prediction and interpersonal beliefs

In N_1-N_2 there is no distinction between decisions and predictions

- E_1 occurs in the scope of $\mathbf{B}_1(\cdot)$
- E_2 occurs in the scope of $\mathbf{B}_1\mathbf{B}_2(\cdot)$

Derivation using N_1-N_2 requires the following infinite regress (from player 1's perspective):

$\mathbf{B}_1(N_1)$		$\mathbf{B}_1\mathbf{B}_2\mathbf{B}_1(N_1)$		$\cdots\cdots$
$\downarrow$	$\nearrow$	$\downarrow$	$\nearrow$	$\downarrow$
$\mathbf{B}_1\mathbf{B}_2(N_2)$		$\mathbf{B}_1\mathbf{B}_2\mathbf{B}_1\mathbf{B}_2(N_2)$		$\cdots\cdots$