# Consumer Choice with N Commodities

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(Lecture 6, Micro Theory I)

# From 2 Goods to N Goods…

- More applications of tools learned in Ch. 1…
- What is needed to…
- Obtain the compensated law of demand?
- Have a concave minimized expenditure function?
- Recover consumer’s demand?
- “Use” a representative agent (in macro)?

# Key Problems to Consider

- Revealed Preference: Only assumption needed:
  - Compensated Law of Demand
  - Concave Minimized Expenditure Function
- Indirect Utility Function: (The Maximized Utility)
  - Roy's Identity: Can recover demand function from it
- Homothetic Preferences: Radial Parallel…
  - Demand is proportional to income
  - Utility function is homogeneous of degree 1
  - Group demand as if one representative agent

# Why do we care about this?

- Three separate questions:
- How general can revealed preference be?
- How do we back out demand from utility maximization?
- When can we aggregate group demand with a representative agent (say in macro)?
- Are these convincing?

# Proposition 2.3-1
# Compensated Price Change

Consider the dual consumer problem

$$M(p,U^*) = \min_x \{p \cdot x | U(x) \geq U^*\}$$

For $x^0$ be expenditure minimizing for prices $p^0$

$x^1$ be expenditure minimizing at prices $p^1$

$x^0, x^1$ satify $U(x) \geq U^*$

$\Rightarrow$ compensated price change is $\Delta p \cdot \Delta x \leq 0$

# Proposition 2.3-1 Compensated Price Change

Proof:

$$p^0 \cdot x^0 \le p^0 \cdot x^1, \quad p^1 \cdot x^1 \le p^1 \cdot x^0$$

Since $x^0$ be expenditure minimizing for prices $p^0$

$x^1$ be expenditure minimizing at prices $p^1$

$$-p^0 \cdot (x^1 - x^0) \le 0, \quad p^1 \cdot (x^1 - x^0) \le 0$$

$$\Rightarrow \Delta p \cdot \Delta x = (p^1 - p^0) \cdot (x^1 - x^0) \le 0$$

# Proposition 2.3-1
# Compensated Price Change

- This is true for any pair of price vectors
- For $p^0 = (\overline{p}_1, \cdots, \overline{p}_{j-1}, p_j^0, \overline{p}_{j+1}, \cdots, \overline{p}_n)$

$$p^1 = (\overline{p}_1, \cdots, \overline{p}_{j-1}, p_j^1, \overline{p}_{j+1}, \cdots, \overline{p}_n)$$

- We have the (compensated) law of demand:

$$\Delta p_j \cdot \Delta x_j \leq 0$$

- Note that we did not need differentiability to get this, just “revealed preferences”!!
- But if that’s true, we do have $\dfrac{\partial x_j^c}{\partial p_j} \leq 0$

# First and Second Derivatives of the Expenditure Function

But what is $\frac{\partial x_j^c}{\partial p_j}$?

Consider the dual problem as a maximization:

$$-M(p, U^*) = \max_x \{-p \cdot x | U(x) \geq U^*\}$$

Lagrangian is $\mathfrak{L} = -p \cdot x + \lambda(U(x) - U^*)$

Envelope Theorem yields $-\frac{\partial M}{\partial p_j} = \frac{\partial \mathfrak{L}}{\partial p_j} = -x_j^c$

$$\Rightarrow \frac{\partial}{\partial p_i}\left(\frac{\partial M}{\partial p_j}\right) = \frac{\partial x_j^c}{\partial p_i}$$

# First and Second Derivatives of the Expenditure Function

Hence, compensated law of demand yields

$$\frac{\partial x_j^c}{\partial p_j} = \frac{\partial^2 M}{\partial p_j^2} \leq 0$$

$\Rightarrow$ Expenditure function concave for each $p_j$.

<u>Is the entire Expenditure function concave?</u>

Requires the matrix of second derivatives

$$\left[\frac{\partial^2 M}{\partial p_i \partial p_j}\right] = \left[\frac{\partial x_j^c}{\partial p_i}\right]$$ to be negative semi-definite

# Proposition 2.3-2
# Concave Expenditure Function

$M(p, U^*)$ is a concave function over $p$.

i.e. For any $p^0, p^1$,

$$M(p, U^*) \geq (1-\lambda)M(p^0, U^*) + \lambda M(p^1, U^*)$$

We can show this with only revealed preferences...
(even without assuming differentiability!)

# Proposition 2.3-2
# Concave Expenditure Function

Proof: For any $x^\lambda$, feasible,

$$M(p^0, U^*) = p^0 \cdot x^0 \leq p^0 \cdot x^\lambda,$$
$$M(p^1, U^*) = p^1 \cdot x^1 \leq p^1 \cdot x^\lambda$$

Since $M(p, U^*)$ minimizes expenditure.

Hence,

$$(1-\lambda)M(p^0, U^*) + \lambda M(p^1, U^*)$$
$$\leq \left[(1-\lambda)p^0 \cdot x^\lambda\right] + \left[\lambda p^1 \cdot x^\lambda\right]$$
$$= p^\lambda \cdot x^\lambda = M(p^\lambda, U^*)$$

# What Have We Learned?

- Method of Revealed Preferences
- Used it to obtain:

1. Compensated Price Change
2. Compensated Law of Demand
3. Concave Expenditure Function
   - Special Case assuming differentiability

- Next: How can we get demand from utility?

# Indirect Utility Function

Let demand for consumer $U(\cdot)$ with income $I$, facing price vector $p$ be $x^* = x(P, I)$.

$$V(p, I) = \min_{x} \{U(x) | p \cdot x \leq I, x \geq 0\}$$
$$= U(x^*(p, I))$$

is maximized $U(x)$, aka **indirect utility function**

Why should we care about this function?

# Proposition 2.3-3
# Roy's Identity

$$x_j^*(p, I) = -\frac{\frac{\partial V}{\partial p_j}}{\frac{\partial V}{\partial I}}$$

Get this directly from indirect utility function...

# Proposition 2.3-3
# Roy's Identity

Proof:

$$V(p,I) = \min_x \{U(x) | p \cdot x \le I, x \ge 0\}$$

Lagrangian is $\mathfrak{L}(x,\lambda) = U(x) + \lambda(I - p \cdot x)$

Envelope Theorem yields $\dfrac{\partial V}{\partial I} = \dfrac{\partial \mathfrak{L}}{\partial I}(x^*, \lambda^*) = \lambda^*$

And $\dfrac{\partial V}{\partial p_j} = \dfrac{\partial \mathfrak{L}}{\partial p_j}(x^*, \lambda^*) = -\lambda^* x_j^*(p,I)$

$$\Rightarrow x_j^*(p,I) = -\frac{\frac{\partial V}{\partial p_j}}{\frac{\partial V}{\partial I}}$$

# Example: Unknown Utility…

Consider indirect utility function

$$V(p, I) = \prod_{i=1}^{n} \left( \frac{\alpha_i I}{p_i} \right)^{\alpha_i} \quad \text{where} \quad \sum_{i=1}^{n} \alpha_i = 1$$

What's the demand (and original utility) function?

$$\ln V = \ln I - \sum_{i=1}^{n} \alpha_i \ln p_i + \sum_{i=1}^{n} \alpha_i \ln \alpha_i$$

$$\Rightarrow \frac{\partial}{\partial I} \ln V = \frac{1}{V} \frac{\partial V}{\partial I} = \frac{1}{I}, \qquad \frac{\partial}{\partial p_i} \ln V = \frac{1}{V} \frac{\partial V}{\partial p_i} = -\frac{\alpha_i}{p_i}$$

# Example: Unknown Utility…

$$V(p, I) = \prod_{i=1}^{n} \left( \frac{\alpha_i I}{p_i} \right)^{\alpha_i} \quad \text{where} \quad \sum_{i=1}^{n} \alpha_i = 1$$

What's the demand (and original utility) function?

$$\ln V = \ln I - \sum_{i=1}^{n} \alpha_i \ln p_i + \sum_{i=1}^{n} \alpha_i \ln \alpha_i$$

$$\Rightarrow \frac{\partial}{\partial I} \ln V = \frac{1}{V} \frac{\partial V}{\partial I} = \frac{1}{I}, \quad \frac{\partial}{\partial p_i} \ln V = \frac{1}{V} \frac{\partial V}{\partial p_i} = -\frac{\alpha_i}{p_i}$$

By Roy's Identity, $x_i^* = -\dfrac{\frac{\partial V}{\partial p_j}}{\frac{\partial V}{\partial I}} = \dfrac{\alpha_i I}{p_i}$

# Example: Cobb-Douglas Utility

- Plugging back in

$$U(x) = V = \prod_{i=1}^{n} \left( \frac{\alpha_i I}{p_i} \right)^{\alpha_i} = \prod_{i=1}^{n} (x_i)^{\alpha_i}$$

- What is this utility function?
- Cobb-Douglas!

- Note: This is an example where demand is proportion to income. In fact, we have…

# Definition: Homothetic Preferences

Strictly monotonic preference $\succsim$ is **homothetic** if, for any $\theta > 0$ and $x^0, x^1$ such that $x^0 \succsim x^1$,

$$\theta x^0 \succsim \theta x^1$$

In fact, if $x^0 \sim x^1$,

Then, $\theta x^0 \sim \theta x^1$

# Why Do We Care About This?

- Proposition 2.3-4:
  - Demand proportional to income
- Proposition 2.3-5:
  - Homogeneous functions represent homothetic preferences
- Proposition 2.3-6:
  - Homothetic preferences are represented by functions that are homogeneous of degree 1
- Proposition 2.3-7: Representative Agent

# Proposition 2.3-4: Demand Proportional to Income

If preferences are homothetic,

and $x^*$ is optimal given income $I$,

Then $\theta x^*$ is optimal given income $\theta I$.

Proof:

Let $x^{**}$ be optimal given income $\theta I$,

Then $x^{**} \succsim \theta x^*$ since $\theta x^*$ is feasible.

By revealed preferences, $x^* \succsim \dfrac{1}{\theta} x^{**}$ ($\frac{1}{\theta} x^{**}$ feasible)

By homotheticity, $\theta x^* \succsim x^{**}$

Thus, $\theta x^* \sim x^{**}$ (optimal for income $\theta I$)

# Proposition 2.3-5: Homogeneous Functions → Homothetic Preferences

If preferences are represented by $U(\lambda x) = \lambda^k U(x)$,

Then preferences are homothetic.

Proof:

Suppose $x \succsim y$,

Then $U(x) \geq U(y)$.

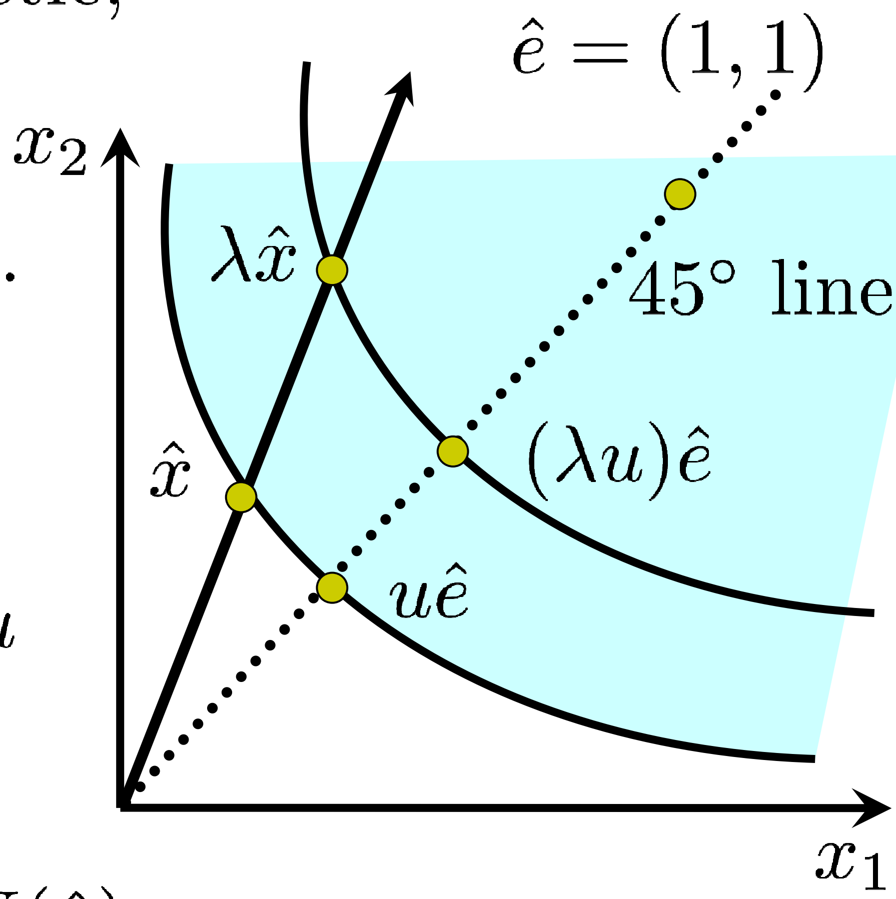
Since $U(x)$ is homogeneous,

$$U(\lambda x) = \lambda^k U(x) \geq \lambda^k U(y) = U(\lambda y)$$

Thus, $\lambda x \succsim \lambda y$ i.e. Preferences are homothetic.

# Proposition 2.3-6: Representation of Homothetic Preferences

If preferences are homothetic,
They can be represented
by a function that is
homogeneous of degree 1.




Proof: $\hat{e} = (1, \cdots, 1)$

For $\hat{x}$, exists $u\hat{e} \sim \hat{x}$

Utility function $U(x) = u$

By homotheticity,

$$\lambda\hat{x} \sim (\lambda u)\hat{e}$$

Hence, $U(\lambda\hat{x}) = \lambda u = \lambda U(\hat{x})$

# Proposition 2.3-7: Representative Preferences

If a group of consumers have the same homothetic preferences,

Then group demand is equal to demand of a representative member holding all the income.

Proof:

Suppose Alex and Bev have the same homothetic preferences, and same demand $x^h = x(p, I^h)$.

By Prop. 2.3-4, $x^A = I^A x(p,1)$, $x^B = I^B x(p,1)$.

$$\Rightarrow x^A + x^B = (I^A + I^B)x(p,1)$$
$$= x(p, I^A + I^B) \text{ by homotheticity}$$

# Summary of 2.3

- Revealed Preference:
  - Compensated Law of Demand
  - Concave Minimized Expenditure Function
- Indirect Utility Function:
  - Roy's Identity: Recovering demand function
- Homothetic Preferences:
  - Demand is proportional to income
  - Utility function is homogeneous of degree 1
  - Group demand as if one representative agent
- Homework: Exercise 2.3-1~5