General Equilibrium for the Exchange Economy

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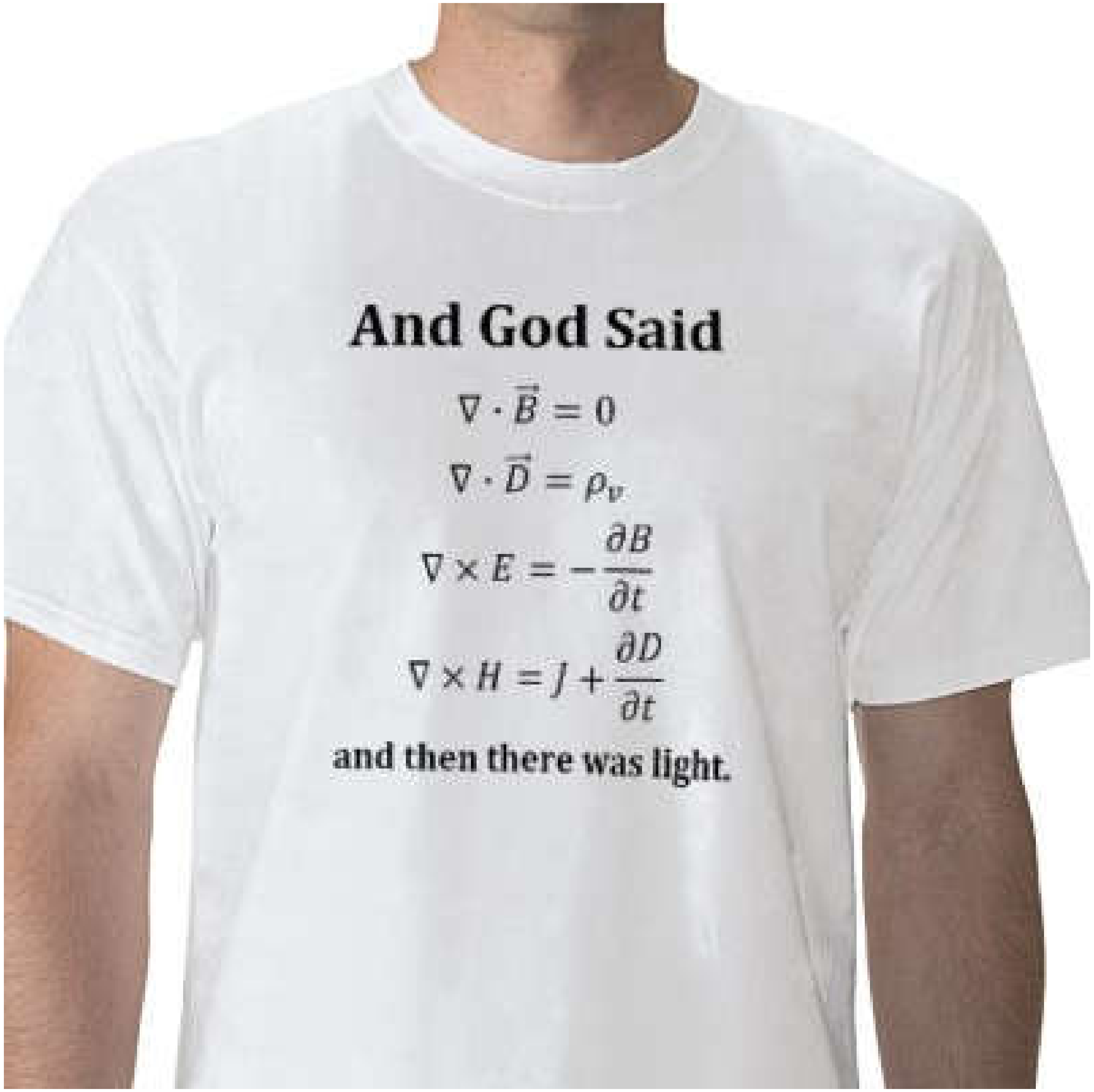
(Lecture 9, Micro Theory I)

What's in between the lines?

- And God said,
  - Let there be light...
- and there was light.... (Genesis 1:3, KJV)

What's in between the lines?



What's in

and God said,

$$E = hf = hc/\lambda,\quad eV_0 = hf - W,\ E = mc^2,\ E^2 = P^2c^2 + m^2c^4,\ \Psi(x,t) = \int_{-\infty}^{\infty} A(k)e^{i(kx-\omega t)}dk,$$

$$p = h/\lambda,\ \Psi(x,t) = e^{i(k_0x-\omega_0 t)}\int_{-\infty}^{\infty} A(k)e^{i(k-k_0)[x-(d\omega/dk)_0 t]}dk,\ v = \left(\frac{d\omega}{dk}\right)_{k_0},\ E = p^2/2m,$$

$$\Psi(x,t) = e^{i(k_0x-\omega_0 t)}\int_{-\infty}^{\infty} A(k)e^{i(k-k_0)[x-(d\omega/dk)_0 t]}dk,\ v = \left(\frac{d\omega}{dk}\right)_{k_0}\quad \hbar\omega e^{i(kx-\omega t)} = \frac{\hbar^2k^2}{2m}e^{i(kx-\omega t)}$$

$$E = \hbar^2k^2/2m,\quad E = \hbar\omega = \hbar^2k^2/2m,\ m_m = \frac{m}{\sqrt{1-v^2/c^2}},\quad \frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} = i\hbar\frac{\partial\Psi}{\partial t}$$

$$\frac{\partial^2\psi}{\partial x^2} + \frac{2m(E-V)}{\hbar^2}\psi = 0,\quad k^2 = \frac{2m(E-V)}{\hbar^2},\quad \lambda = \frac{h}{\sqrt{2m(E-V)}},\quad E = \frac{1}{2}kx^2$$

$$E\psi = -\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right) - \frac{2e^2}{4\pi\varepsilon_0 r}\psi,\quad J = \nabla\times H,\quad \frac{d^2x}{dt^2} + \frac{k}{x}x = 0$$

$$J = \frac{1}{r\sin\theta}\left[\frac{\partial H_\phi\sin\theta}{\partial\theta} - \frac{\partial H_\theta}{\partial\phi}\right]\bar{a}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial H_r}{\partial\phi} - \frac{\partial(rH_\phi)}{\partial r}\right]\bar{a}_\theta + \frac{1}{r}\left[\frac{\partial(rH_\theta)}{\partial r} - \frac{\partial H_r}{\partial\theta}\right]\bar{a}_\phi$$

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right) + V\psi = E\psi,\quad V = -\frac{e^2}{4\pi\varepsilon_0}\frac{1}{r} = \frac{e^2}{4\pi\varepsilon_0}\frac{1}{\sqrt{x^2+y^2+z^2}}$$

$$\nabla^2 v = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v}{\partial\phi^2},\quad J = \lim_{\Delta S_n\to 0}\frac{\oint H\cdot dl}{\Delta S_n}$$

$$\nabla\cdot D = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u}(h_2h_3D_u) + \frac{\partial}{\partial v}(h_3h_1D_v) + \frac{\partial}{\partial w}(h_1h_2D_w)\right]$$

$$P_d = \int_{vol}\frac{1}{\sigma}J^2dv = \int_{-l}^{0}\int_0^{2\pi}\int_a^{b}\frac{4\sigma V_0^2}{\left[r\ln\left(b/a\right)\right]^2}\sin^2\beta z\sin^2\omega t\, r\,dr\,d\theta\,dz = \frac{4\pi\sigma V_0^2}{\ln\left(b/a\right)}\left(l - \frac{\sin 2\beta l}{2\beta}\right)\sin^2\omega t$$

$$J_v(z) = \sum_{m=0}^{\infty}\frac{(-1)^m z^{v+2m}}{m!\,\Gamma(m+v+1)2^{v+2m}},\quad J_{-v}(z) = \sum_{m=0}^{\infty}\frac{(-1)^m z^{-v+2m}}{m!\,\Gamma(m-v+1)2^{-v+2m}}$$

$$\oint_l\bar{E}\cdot d\bar{l} = emf = -\int_s\frac{\partial\bar{B}}{\partial t}\cdot ds,\quad \oint_l\bar{H}\cdot d\bar{l} = I = \int\left(\bar{J}_c + \frac{\partial\bar{D}}{\partial t}\right)\cdot ds,\quad \oint_s\bar{D}\cdot d\bar{S} = Q = \int_v\nabla\cdot\bar{D}dv$$

$$E_r = \frac{J_0e^{-\gamma r}}{4\pi}\left(\sqrt{\frac{\mu}{\varepsilon}}\frac{2}{r^2} + \frac{2}{j\omega\varepsilon r^3}\right)\cos\theta,\quad E_\theta = \frac{J_0e^{-\gamma r}}{4\pi}\left(\frac{j\omega\mu}{r} + \sqrt{\frac{\mu}{\varepsilon}}\frac{1}{r^2} + \frac{1}{j\omega\varepsilon r^3}\right)\sin\theta$$

$$E(r,\theta,t) = \frac{-\omega_0\mu J_0}{4\pi r}\sin\theta\sin(\omega t - \omega r\sqrt{\mu\varepsilon})a_\theta,\quad H(r,\theta,t) = \sqrt{\frac{\varepsilon}{\mu}}E_\theta a_\phi,\quad \gamma = j\omega\sqrt{\mu\varepsilon}\ \ldots$$

and there was light.

What We Learned from the 2x2 Economy?

- **Pareto Efficient Allocation (PEA)**
  - **Cannot** make one better off without hurting others
- **Walrasian Equilibrium (WE)**
  - When Supply Meets Demand
  - Focus on Exchange Economy First
- 1st Welfare Theorem: WE is Efficient
- 2nd Welfare Theorem: Any PEA can be supported as a WE
- These also apply to the general case as well!

General Exchange Economy

- n Commodities: *1, 2, ..., n*
- H Consumers: $h = 1, 2, \cdots, H$
  - Consumption Set: $X^h \subset \mathbb{R}^n_+$
  - Endowment: $\vec{\omega}^h = (\omega_1^h, \cdots, \omega_n^h) \in X^h$
  - Consumption Vector: $\vec{x}^h = (x_1^h, \cdots, x_n^h) \in X^h$
  - Utility Function: $U^h(\vec{x}^h) = U^h(x_1^h, \cdots, x_n^h)$
  - Aggregate Consumption and Endowment:
    $\vec{x} = \sum_{h=1}^{H} \vec{x}^h$ and $\vec{\omega} = \sum_{h=1}^{H} \vec{\omega}^h$
- Edgeworth Cube (Hyperbox)

Feasible Allocation

- A allocation is feasible if
- The sum of all consumers' demand doesn't exceed aggregate endowment: $\vec{x} - \vec{\omega} \leq \vec{0}$
- A feasible allocation $\vec{\mathbb{x}}$ is Pareto efficient if
- there is no other feasible allocation $\vec{x}$ that is
- strictly preferred by at least one: $U^i(\vec{x}^i) > U^i(\vec{\mathbb{x}}^i)$
- and is weakly preferred by all: $U^h(\vec{x}^h) \geq U^h(\vec{\mathbb{x}}^h)$

Walrasian Equilibrium

- Price-taking: Price vector $\vec{p} \geq \vec{0}$
- Consumers: $h=1,\ 2,\ \ldots,\ H$
- Endowment: $\vec{\omega}^h = (\omega_1^h, \cdots, \omega_n^h) \quad \vec{\omega} = \sum_h \vec{\omega}^h$
- Wealth: $W^h = \vec{p} \cdot \vec{\omega}^h$
- Budget Set: $\{\vec{x}^h \in X^h | \vec{p} \cdot \vec{x}^h \leq W^h\}$
- Consumption Set: $\vec{\mathbb{x}}^h = (\mathbb{x}_1^h, \cdots, \mathbb{x}_n^h) \in X^h$
- Most Preferred Consumption:
  $U^h(\vec{\mathbb{x}}^h) \geq U^h(\vec{x}^h)$ for all $\vec{x}^h$ such that $\vec{p} \cdot \vec{x}^h \leq W^h$
- Vector of Excess Demand: $\vec{e} = \vec{\mathbb{x}} - \vec{\omega}$

Definition: Walrasian Equilibrium Prices

- The price vector $\vec{p} \geq \vec{0}$ is a Walrasian Equilibrium price vector if
- there is no market in excess demand ($\vec{e} \leq \vec{0}$),
- and $p_j = 0$ for any market that is in excess supply ($e_j < 0$).

- We are now ready to state and prove the "Adam Smith Theorem" (WE $\Rightarrow$ PEA)...

Proposition 3.2-0: First Welfare Theorem

- If preferences of each consumer satisfies LNS, then the Walrasian Equilibrium allocation is Pareto efficient.

- Proof:
  - (Same as 2-consumer case. Homework.)

SWT without differentiability

- In Section 3.1, we assumed differentiability to use Kuhn-Tucker conditions to prove SWT
- Now we drop differentiability and appeal directly to Supporting Hyperplane Theorem
- To do that, we first need a lemma...

Lemma 3.2-1: Quasi-concavity of V

- If $U^h, h = 1, \cdots, H$ is quasi-concave,
- Then so is the indirect utility function

$$V^1(\vec{x}) = \max_{\vec{x}^h} \left\{ U^1(\vec{x}^1) \middle| \sum_{h=1}^{H} \vec{x}^h \leq \vec{x}, \right.$$

$$\left. U^h(\vec{x}^h) \geq U^h(\vec{\mathbb{x}}^h), h \neq 1 \right\}$$

Lemma 3.2-1: Quasi-concavity of V

- Proof: For aggregate endowment $\vec{a}, \vec{b}$, claim for $\vec{c} = (1-\lambda)\vec{a} + \lambda\vec{b}$, $V^1(\vec{c}) \geq \min\{V^1(\vec{a}), V^1(\vec{b})\}$

Assume $\{\vec{a}^h\}_{h=1}^H$ solves $V^1(\vec{a}) = U^1(\vec{a}^1)$

$\{\vec{b}^h\}_{h=1}^H$ solves $V^1(\vec{b}) = U^1(\vec{b}^1)$

$\{\vec{c}^h\}_{h=1}^H$ is feasible since $\vec{c}^h = (1-\lambda)\vec{a}^h + \lambda\vec{b}^h$

$\Rightarrow V^1(\vec{c}) \geq U^1(\vec{c}^1)$

Now only need to prove $U^1(\vec{c}^1) \geq \min\{V^1(\vec{a}), V^1(\vec{b})\}$.

Lemma 3.2-1: Quasi-concavity of V

- Since $\{\vec{a}^h\}_{h=1}^H$ solves $V^1(\vec{a})$,

  $\{\vec{b}^h\}_{h=1}^H$ solves $V^1(\vec{b})$,

  $U^1(\vec{a}^1) = V^1(\vec{a})$ and $U^1(\vec{b}^1) = V^1(\vec{b})$

  by quasi-concavity of U^1

$$\Rightarrow U^1(\vec{c}^1) \geq \min\{U^1(\vec{a}^1), U^1(\vec{b}^1)\}$$

$$= \min\{V^1(\vec{a}), V^1(\vec{b})\}$$

$$\Rightarrow V^1(\vec{c}) \geq U^1(\vec{c}^1) \geq \min\{V^1(\vec{a}), V^1(\vec{b})\}$$

Proposition 3.2-2: Second Welfare Theorem

- Consumer $h \in \mathcal{H}$ has endowment $\vec{\omega}^h \in \mathbb{R}^n_+$
- Suppose $X^h = \mathbb{R}^n_+$, and utility functions $U^h(\cdot)$
- continuous, quasi-concave, strictly monotonic.
- If $\{\vec{\mathbb{x}}^h\}_{h=1}^H$ where $\vec{\mathbb{x}}^h \neq \vec{0}$ is Pareto efficient,
- then there exist a price vector $\vec{p} \gg \vec{0}$ such that

$$U^h(\vec{x}^h) > U^h(\vec{\mathbb{x}}^h) \Rightarrow \vec{p} \cdot \vec{x}^h > \vec{p} \cdot \vec{\mathbb{x}}^h$$

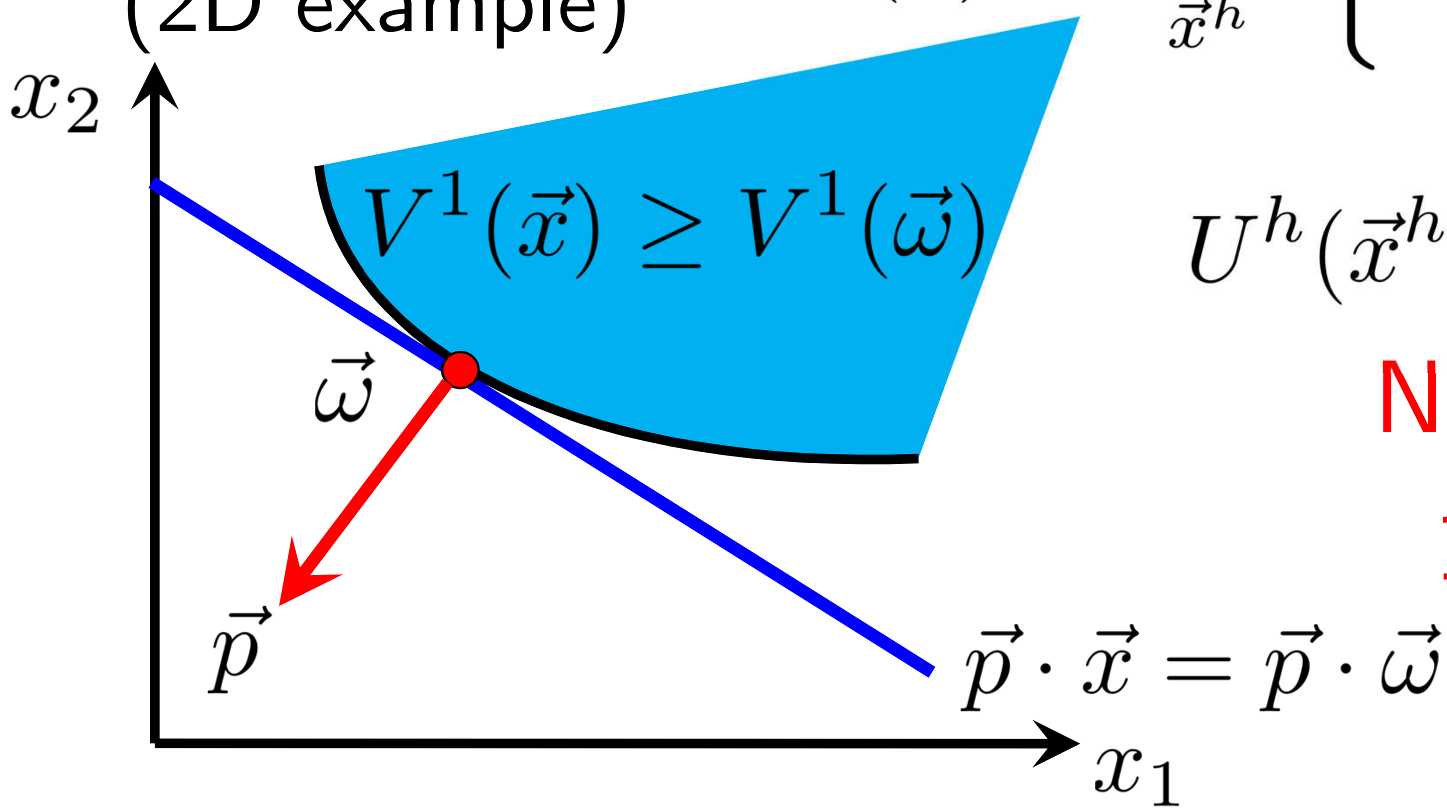
- Proof:

Proposition 3.2-2: Second Welfare Theorem

- Proof: Want to apply Supporting Hyperplane Theorem to the set $\{\vec{x} | V^1(\vec{x}) \geq V^1(\vec{\omega})\}$ where

$$V^1(\vec{x}) = \max_{\vec{x}^h} \left\{ U^1(\vec{x}^1) \middle| \sum_{h=1}^{H} \vec{x}^h \leq \vec{x}, \; U^h(\vec{x}^h) \geq U^h(\vec{\mathbb{x}}^h), h \neq 1 \right\}$$

(2D example)



Need to show that:

1. $\vec{\omega}$ on boundary
2. Set is convex

Proposition 3.2-2: Second Welfare Theorem

- Proof: Assume nobody has zero allocation
  - Relaxing this is easily done...
- By Lemma 3.2-1, $V^1(\vec{x})$ is quasi-concave
  - Convex upper contour set $\{\vec{x} | V^1(\vec{x}) \geq V^1(\vec{\omega})\}$
- $V^1(\vec{x})$ is strictly increasing since $U^1(\cdot)$ is also
  - and any increment could be given to consumer *1*
- Since $\{\vec{\mathbb{x}}^h\}_{h=1}^H$ is Pareto efficient, $V^1(\vec{\omega}) = U^1(\vec{\mathbb{x}}^1)$
- Since $U^1(\cdot)$ is strictly increasing, $\sum_{h=1}^{H} \vec{\mathbb{x}}^h = \vec{\omega}$

Proposition 3.2-2: Second Welfare Theorem

- Proof (Continued):
- Since $\vec{\omega}$ is on the boundary of $\{\vec{x}|V^1(\vec{x}) \geq V^1(\vec{\omega})\}$
- By the Supporting Hyperplane Theorem, there exists a vector $\vec{p} \neq \vec{0}$ such that

$$V^1(\vec{x}) > V^1(\vec{\omega}) \Rightarrow \vec{p} \cdot \vec{x} > \vec{p} \cdot \vec{\omega}$$

$$\text{and } V^1(\vec{x}) \geq V^1(\vec{\omega}) \Rightarrow \vec{p} \cdot \vec{x} \geq \vec{p} \cdot \vec{\omega}$$

- Claim: $\vec{p} \gg \vec{0}$, then we can show that

$$U^h(\vec{x}^h) > U^h(\vec{\mathbb{x}}^h) \Rightarrow \vec{p} \cdot x^h > \vec{p} \cdot \vec{\mathbb{x}}^h$$

Proposition 3.2-2: Second Welfare Theorem

- Proof (Continued):
- Why $\vec{p} \gg \vec{0}$? If not, define $\vec{\delta} = (\delta_1, \cdots, \delta_n) > 0$
- such that $\delta_j > 0$ iff $p_j < 0$ (others $= 0$)
- Then, $V^1(\vec{\omega} + \vec{\delta}) > V^1(\vec{\omega})$ and $\vec{p} \cdot (\vec{\omega} + \vec{\delta}) < \vec{p} \cdot \vec{\omega}$
- Contradicting (Supporting Hyperplane Thm)

$$U^h(\vec{x}^h) \geq U^h(\vec{\mathbb{x}}^h) \Rightarrow \vec{p} \cdot \sum_{h=1}^{H} \vec{x}^h \geq \vec{p} \cdot \vec{\omega}$$

$$V^1(\vec{x}) > V^1(\vec{\omega}) \Rightarrow \vec{p} \cdot \sum_{h=1} \vec{x}^h > \vec{p} \cdot \vec{\omega}$$

Proposition 3.2-2: Second Welfare Theorem

- Since $U^h(\vec{x}^h) \geq U^h(\vec{\mathbb{x}}^h) \Rightarrow \vec{p} \cdot \sum_{h=1}^{H} \vec{x}^h \geq \vec{p} \cdot \sum_{h=1}^{H} \vec{\mathbb{x}}^h$
- Set $\vec{x}^k = \vec{\mathbb{x}}^k$ for all $k \neq h$, then for consumer h

$$U^h(\vec{x}^h) \geq U^h(\vec{\mathbb{x}}^h) \Rightarrow \vec{p} \cdot \vec{x}^h \geq p \cdot \vec{\mathbb{x}}^h$$

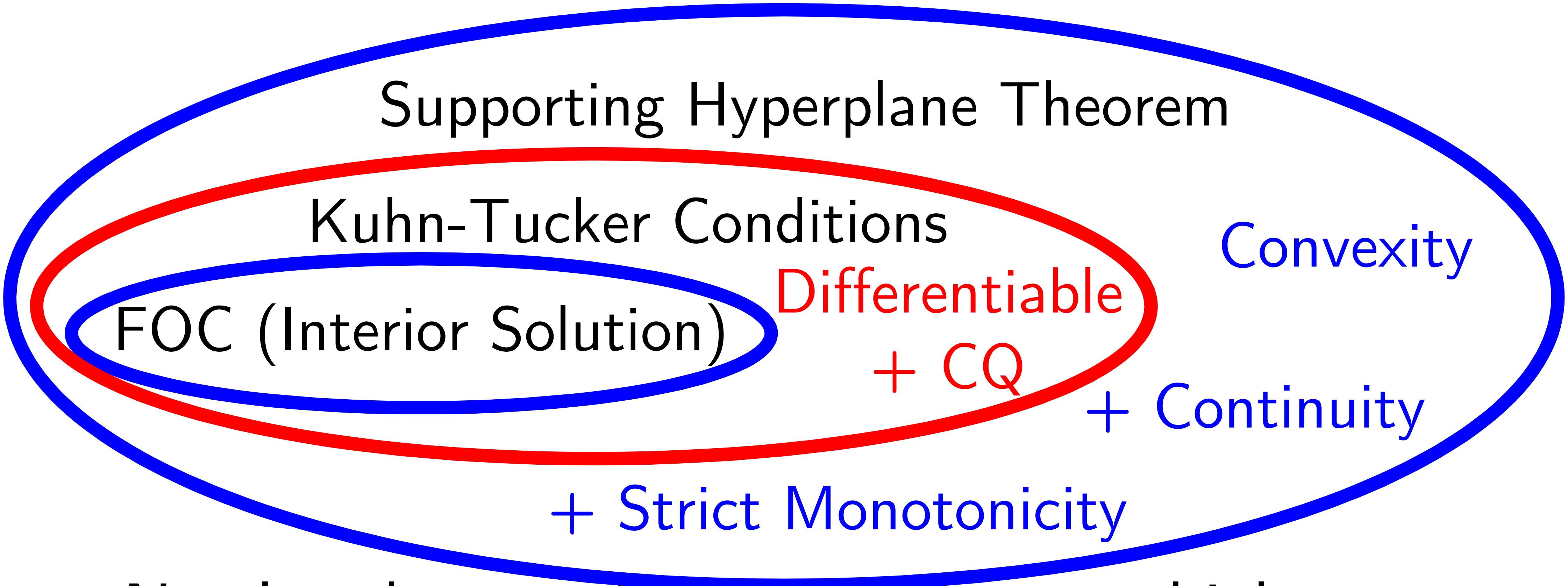
- Need to show strict inequality implies strict…
- If not, then $U^h(\vec{x}^h) > U^h(\vec{\mathbb{x}}^h) \Rightarrow \vec{p} \cdot \vec{x}^h = \vec{p} \cdot \vec{\mathbb{x}}^h$
- Hence, $\vec{p} \cdot \lambda \vec{x}^h < \vec{p} \cdot \vec{\mathbb{x}}^h \text{ for all } \lambda \in (0,1)$

$$U^h \text{ continuous } \Rightarrow U^h(\lambda \vec{x}^h) > U^h(\vec{\mathbb{x}}^h) \text{ for } \lambda \text{ near } 1$$

- Contradiction!

Why should I care about this (or the math)?

- In Ch.3 we saw three different versions of the SWT, each with different assumptions...



- Need to know when can you use which...

Summary of 3.2

- Pareto Efficiency:
  - Cannot make one better off without hurting others
- Walrasian Equilibrium: market clearing prices
- Welfare Theorems:
  - First: Walrasian Equilibrium is Pareto Efficient
  - Second: Pareto Efficient allocations can be supported as Walrasian Equilibria (with transfer)
- Homework: Prove FWT for n-consumers
  - (Optional: 2009 final-Part B)