Budget Constrained Choice with Two Commodities

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(Lecture 5, Micro Theory I)

The Consumer Problem

- We have some powerful tools:
  - Constrained Maximization (Shadow Prices)
  - Envelope Theorem (Changing Environment)
- Can help us understand consumer behavior?

  Such as:
  - Maximizing utility, facing a budget constraint
  - Minimizing cost, maintaining certain welfare level

Key Problems to Consider

- Total Price Effect =

- Substitution Effect + Income Effect
- Consumer Problem: How can consumer's Utility Maximization result in demand?
  - Income Effect: How does an increase/decrease in income (budget) affect demand?
- Dual Problem: How is Minimizing Expenditure related to Maximizing Utility?
  - Substitution Effect: How does an increase in commodity price affect compensated demand?

Why do we care about this? Public Policy!

- Taiwan's ministry of defense has to decide whether to buy more fighter jets, or more submarines given a tight budget
- How does the military rank each combination?
- How do they choose which combination to buy?
- How would a price change affect their decision?
- How would a boycott in defense budget affect their decision?

Continuous Demand Function

Consumer with income I faces prices $\vec{p} = (p_1, p_2)$

$$\max_{\vec{x}} \{U(\vec{x}) | \vec{p} \cdot \vec{x} \leq I, \vec{x} \in \mathbb{R}^2_+\}$$

- Assume: LNS (local non-satiation)
  - Then, consumer spends all his/her income!
- $U(\vec{x})$ is continuous, strictly quasi-concave on $\mathbb{R}^2_+$
  - There is a unique solution $\vec{x}^0 = \vec{x}(\vec{p}, I)$
- Then, by Prop. 2.2-1, $\vec{x}(\vec{p}, I)$ must be continuous.
  - aka Theory of Maximum I (Prop. C.4-1 on p. 581)

Appendix C:
Prop.C.4-1 Theory of Maximum I

- For f continuous, define

$$F(\vec{\alpha}) = \max_{\vec{x}} \left\{ f(\vec{x},\vec{\alpha}) \middle| \vec{x} \geq 0, \quad \vec{x} \in X(\vec{\alpha}) \subset \mathbb{R}^n, \quad \vec{\alpha} \in A \subset \mathbb{R}^m \right\}$$

- If (i) for each $\vec{\alpha}$ there is a unique

$$\vec{x}^*(\vec{\alpha}) = \arg\max_{\vec{x}} \left\{ f(\vec{x},\vec{\alpha}) \middle| \vec{x} \geq 0, \vec{x} \in X(\vec{\alpha}), \vec{\alpha} \in A \right\}$$

- and (ii) $X(\vec{\alpha})$ is a compact-valued correspondence that is continuous at $\vec{\alpha}^0$
- Then, $\vec{x}^*(\vec{\alpha})$ is continuous at $\vec{\alpha}^0$

Continuous Demand of Prices and Income

- $U(\vec{x})$ is continuous, strictly quasi-concave on $\mathbb{R}^2_+$

$$F(\vec{\alpha}) = \max_{\vec{x}} \{U(\vec{x}) | \vec{p} \cdot \vec{x} \le I, \vec{x} \in \mathbb{R}^2_+\}$$

- If (i) for each $\vec{\alpha}$ there is a unique

$$\vec{x}^0 = \vec{x}(\vec{p}, I)$$

Consumer with income I faces prices $\vec{p} = (p_1, p_2)$

- Then, $\vec{x}(\vec{p}, I)$ must be continuous.

Some Stronger Convenience Assumptions

- Assume:
- $U(\vec{x})$ is continuously differentiable on $\mathbb{R}^2_+$
  - FOC is gradient vector of utility (& constraints)
- LNS-plus:

$$\frac{\partial U}{\partial \vec{x}}(\vec{x}) = \left(\frac{\partial U}{\partial x_1}, \frac{\partial U}{\partial x_2}\right) \gg \vec{0} \text{ for all } \vec{x} \in \mathbb{R}^2_+$$

  - MU > 0: Preferences are strictly increasing
- No corners: $\lim_{x_j \to 0} \frac{\partial U}{\partial x_j} = \infty, j = 1, 2$
  - Always wants to consume some of everything

Indifference Curve Analysis (Lagrangian Ver.)

A Consumer with income I, facing prices p_1, p_2

$$\max_{\vec{x}} \left\{ U(\vec{x}) | \vec{p} \cdot \vec{x} \leq I, \vec{x} \in \mathbb{R}_+^2 \right\}$$

Lagrangian is $\mathfrak{L} = U + \lambda(I - \vec{p} \cdot \vec{x})$

$$(FOC) \qquad \frac{\partial \mathfrak{L}}{\partial x_j} = \frac{\partial U}{\partial x_j}(\vec{x}^*) - \lambda p_j = 0, j = 1, 2$$

$$\frac{\frac{\partial U}{\partial x_1}(\vec{x}^*)}{p_1} = \frac{\frac{\partial U}{\partial x_2}(\vec{x}^*)}{p_2} = \lambda$$

Meaning of FOC

1. Same marginal value for last dollar spent on each commodity $$\frac{\frac{\partial U}{\partial x_1}(\vec{x}^*)}{p_1} = \frac{\frac{\partial U}{\partial x_2}(\vec{x}^*)}{p_2} = \lambda$$
   – Does Taiwan get the same defense MU on fighter jets and submarines?
2. Indifference Curve tangent to Budget Line
$$MRS(\vec{x}^*) = \frac{\frac{\partial U}{\partial x_1}(\vec{x}^*)}{\frac{\partial U}{\partial x_2}(\vec{x}^*)} = \frac{p_1}{p_2}$$

Three Examples

- Quasi-Linear Convex Preference

$$U(\vec{x}) = v(x_1) + \alpha x_2$$

- Cobb-Douglas Preferences

$$U(\vec{x}) = x_1^{\alpha_1} x_2^{\alpha_2}, \alpha_1, \alpha_2 > 0$$

- CES Utility Function

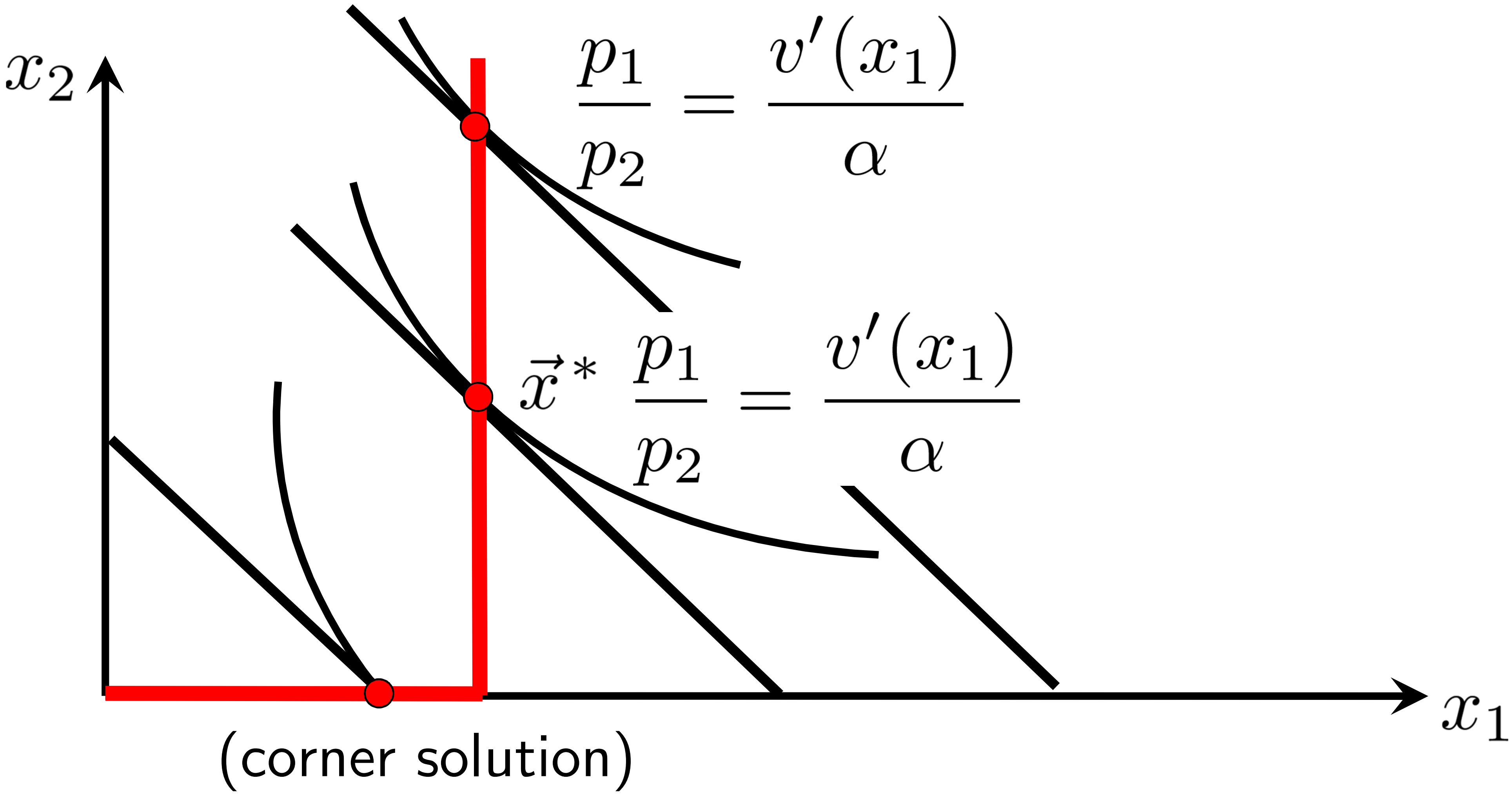
$$U(\vec{x}) = \left(\alpha_1 x_1^{1-\frac{1}{\theta}} + \alpha_2 x_2^{1-\frac{1}{\theta}}\right)^{\frac{1}{1-\frac{1}{\theta}}}$$

Quasi-Linear Convex Utility

$$\max_{\vec{x}} \left\{ U(\vec{x}) = v(x_1) + \alpha x_2 \middle| p_1 x_1 + p_2 x_2 \leq I, x \in \mathbb{R}_+^2 \right\}$$

- FOC: $$\frac{\frac{\partial U}{\partial x_1}}{p_1} = \frac{\frac{\partial U}{\partial x_2}}{p_2} = \frac{v'(x_1)}{p_1} = \frac{\alpha}{p_2} (= \lambda)$$
- Implication: $\frac{p_1}{p_2} = \frac{v'(x_1)}{\alpha}$ (MRS=price)
- Note that x_2 is irrelevant...
- What does this mean?

Income Effect



- Vertical Income Expansion Path (at interior)

Cobb-Douglas Preferences

$$\max_{x_1, x_2} U(x_1, x_2) = x_1^{\alpha_1} x_2^{\alpha_2}, \alpha_1 + \alpha_2 = 1$$

$$\text{s.t. } P_{x_1} \cdot x_1 + P_{x_2} \cdot x_2 \leq I = P_{x_1} \cdot \omega_{x_1} + P_{x_2} \cdot \omega_{x_2}$$

$$\mathcal{L} = x_1^{\alpha_1} x_2^{\alpha_2} + \lambda \cdot [I - P_{x_1} \cdot x_1 - P_{x_2} \cdot x_2]$$

FOC: (for interior solutions)

$$\frac{\partial \mathcal{L}}{\partial x_1} = \alpha_1 \cdot \frac{x_2^{\alpha_2}}{x_1^{\alpha_2}} - \lambda \cdot P_{x_1} = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = \alpha_2 \cdot \frac{x_1^{\alpha_1}}{x_2^{\alpha_1}} - \lambda \cdot P_{x_2} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = I - P_{x_1} \cdot x_1 - P_{x_2} \cdot x_2 = 0$$

Cobb-Douglas Preferences

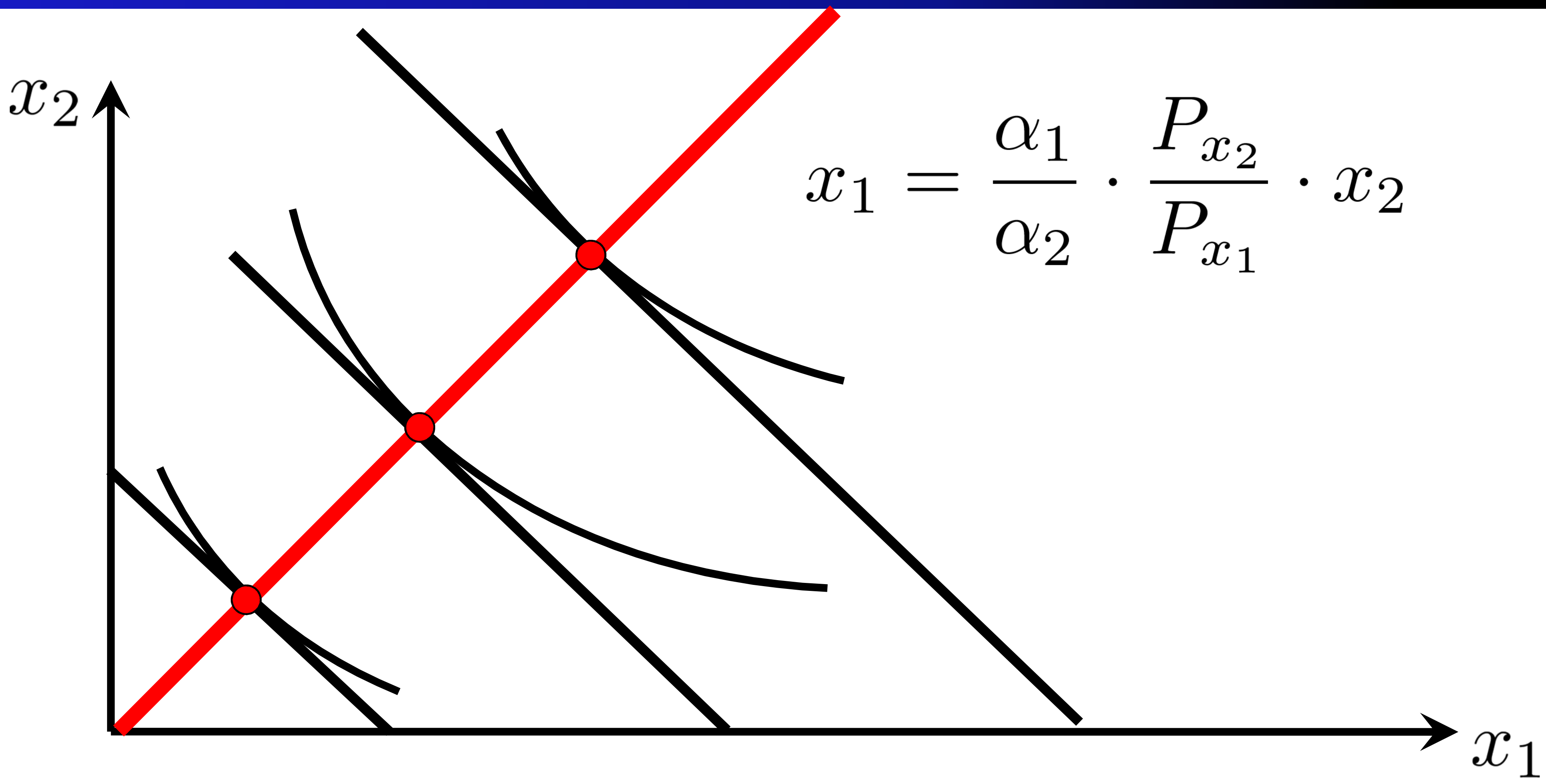
- Meaning of FOC: $MRS = \frac{P_{x_1}}{P_{x_2}}$

$$\frac{P_{x_1}}{P_{x_2}} = \frac{\alpha_1}{\alpha_2} \cdot \frac{x_2}{x_1} \qquad \Rightarrow x_1 = \frac{\alpha_1}{\alpha_2} \cdot \frac{P_{x_2}}{P_{x_1}} \cdot x_2$$

$$\Rightarrow I = P_{x_1} \cdot x_1 + P_{x_2} \cdot x_2 = \frac{\alpha_1 + \alpha_2}{\alpha_2} \cdot P_{x_2} \cdot x_2$$

$$\Rightarrow x_2^* = \frac{\alpha_2}{\alpha_1 + \alpha_2} \cdot \frac{I}{P_{x_2}}, \; x_1^* = \frac{\alpha_1}{\alpha_1 + \alpha_2} \cdot \frac{I}{P_{x_1}}$$

Income Effect



$$x_1 = \frac{\alpha_1}{\alpha_2} \cdot \frac{P_{x_2}}{P_{x_1}} \cdot x_2$$

- Linear Income Expansion Path…

CES Utility Function

$$U(\vec{x}) = \left(\alpha_1 x_1^{1-\frac{1}{\theta}} + \alpha_2 x_2^{1-\frac{1}{\theta}}\right)^{\frac{1}{1-\frac{1}{\theta}}}$$

$$\mathcal{L} = \left(\alpha_1 x_1^{1-\frac{1}{\theta}} + \alpha_2 x_2^{1-\frac{1}{\theta}}\right)^{\frac{1}{1-\frac{1}{\theta}}} + \lambda \cdot \left[I^A - P_x \cdot x - P_y \cdot y\right]$$

- FOC: (for interior solutions)

$$\frac{\partial \mathcal{L}}{\partial x_1} = \alpha_1 x_1^{-\frac{1}{\theta}} \cdot \left(\alpha_1 x_1^{1-\frac{1}{\theta}} + \alpha_2 x_2^{1-\frac{1}{\theta}}\right)^{\frac{1}{\theta-1}} - \lambda \cdot P_{x_1} = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = \alpha_2 x_2^{-\frac{1}{\theta}} \cdot \left(\alpha_1 x_1^{1-\frac{1}{\theta}} + \alpha_2 x_2^{1-\frac{1}{\theta}}\right)^{\frac{1}{\theta-1}} - \lambda \cdot P_{x_2} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = I - P_{x_1} \cdot x_1 - P_{x_2} \cdot x_2 = 0$$

CES Utility Function

$$\frac{P_{x_1}}{P_{x_2}} = \frac{\alpha_1}{\alpha_2} \cdot \left(\frac{x_2}{x_1}\right)^{\frac{1}{\theta}} \Rightarrow x_1 = \left(\frac{\alpha_1}{\alpha_2} \cdot \frac{P_{x_2}}{P_{x_1}}\right)^{\theta} \cdot x_2$$

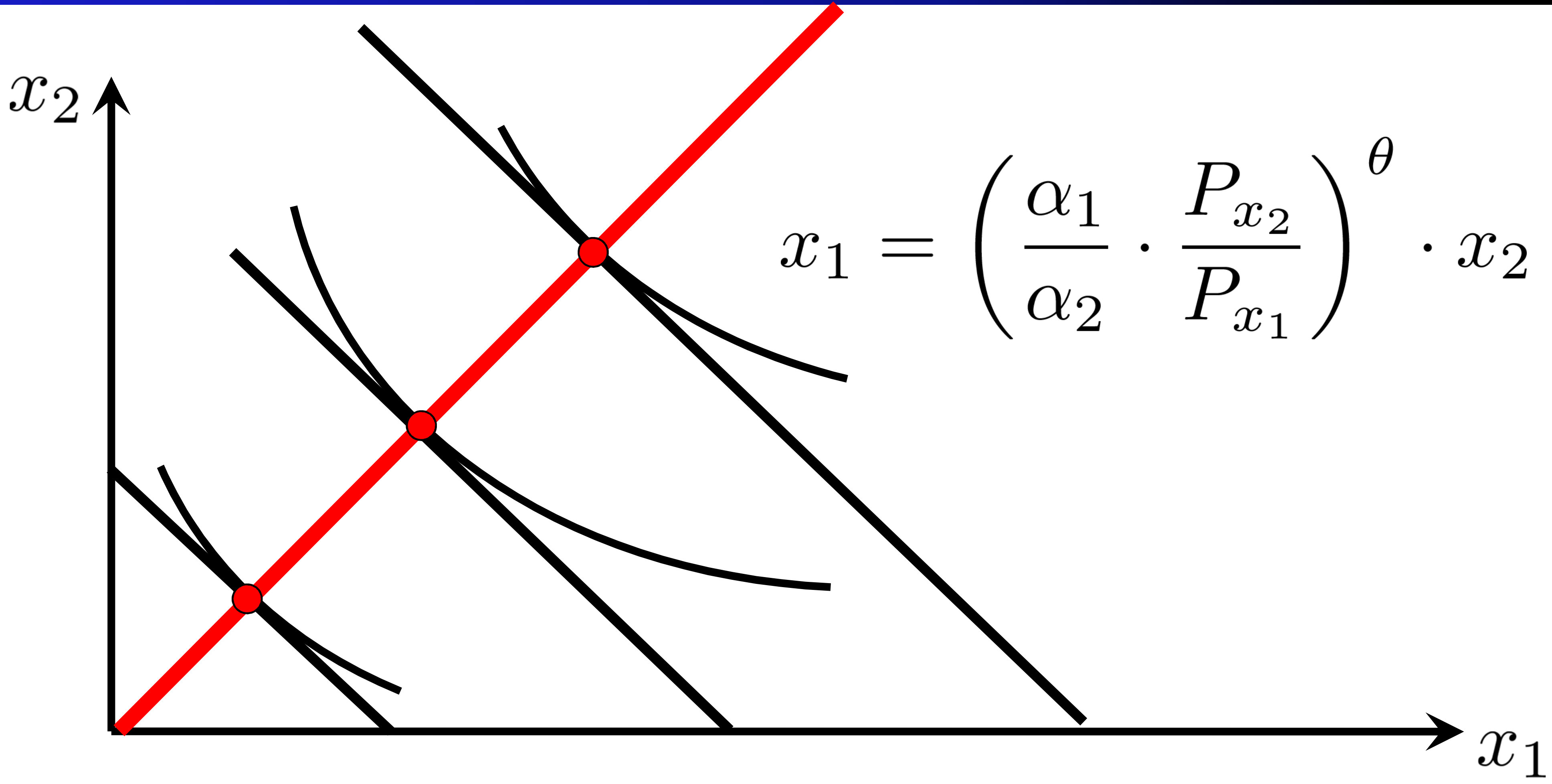
$$\Rightarrow I = P_{x_1} \cdot x_1 + P_{x_2} \cdot x_2$$

$$= \left[\left(\frac{\alpha_1}{\alpha_2}\right)^{\theta} \cdot \left(\frac{P_{x_2}}{P_{x_1}}\right)^{\theta-1} + 1\right] \cdot P_{x_2} \cdot x_2$$

$$\Rightarrow x_2^* = \frac{\alpha_2^{\theta} P_{x_1}^{\theta-1}}{\alpha_1^{\theta} P_{x_2}^{\theta-1} + \alpha_2^{\theta} P_{x_1}^{\theta-1}} \cdot \frac{I}{P_{x_2}},$$

$$x_1^* = \frac{\alpha_1^{\theta} P_{x_1}^{\theta-1}}{\alpha_1^{\theta} P_{x_2}^{\theta-1} + \alpha_2^{\theta} P_{x_1}^{\theta-1}} \cdot \frac{I}{P_{x_1}}$$

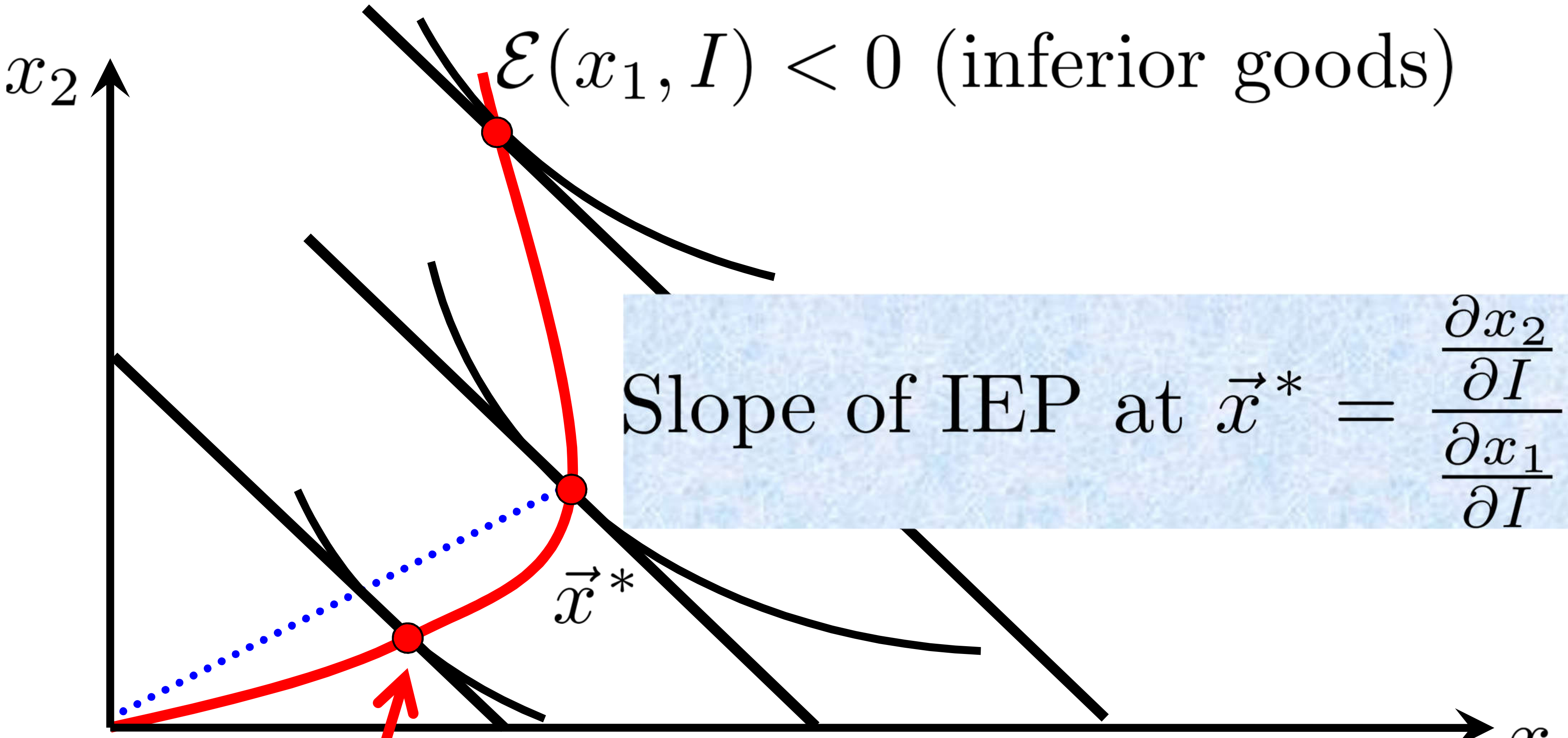
Income Effect



$$x_1 = \left(\frac{\alpha_1}{\alpha_2} \cdot \frac{P_{x_2}}{P_{x_1}}\right)^{\theta} \cdot x_2$$

- Linear Income Expansion Path…
- Cobb-Douglas is a special case of CES! $(\theta = 1)$

Income Effects



$$\text{Slope of IEP at } \vec{x}^* = \frac{\frac{\partial x_2}{\partial I}}{\frac{\partial x_1}{\partial I}} > \frac{x_2^*}{x_1^*}$$

Income Elasticity of Demand

$$\mathcal{E}(x_1, I) = \frac{I}{x_1}\frac{\partial x_1}{\partial I} > 0 \text{ (normal goods)}$$

Income Effects

- If IEP is steeper than the line joining 0 & x^*
- Then, $$\text{Slope of IEP at } \vec{x}^* = \frac{\frac{\partial x_2}{\partial I}}{\frac{\partial x_1}{\partial I}} > \frac{x_2^*}{x_1^*}$$
- Or, $$\mathcal{E}(x_2, I) = \frac{I}{x_2}\frac{\partial x_2}{\partial I} > \mathcal{E}(x_1, I) = \frac{I}{x_1}\frac{\partial x_1}{\partial I}$$
- Lemma 2.2-2: Expenditure share weighted average income elasticity $= 1$
- So, $\mathcal{E}(x_2, I) > 1 > \mathcal{E}(x_1, I)$

Lemma 2.2-2:
Weighted Average Income Elasticity

– (Expenditure-Share Weighted) Average IE = 1

$$k_1\mathcal{E}(x_1^*, I) + k_2\mathcal{E}(x_2^*, I) = 1$$

- Where $k_j = \dfrac{p_j x_j^*}{I}$ is the expenditure share of x_j

Proof:

$$p_1 x_1^*(\vec{p}, I) + p_2 x_2^*(\vec{p}, I) = I \Rightarrow p_1 \frac{\partial x_1^*}{\partial I} + p_2 \frac{\partial x_2^*}{\partial I} = 1$$

$$\Rightarrow \underbrace{\left(\frac{p_1 x_1^*}{I}\right)}_{k_1} \underbrace{\frac{I}{x_1^*}\frac{\partial x_1^*}{\partial I}}_{\mathcal{E}(x_1^*, I)} + \underbrace{\left(\frac{p_2 x_2^*}{I}\right)}_{k_2} \underbrace{\frac{I}{x_2^*}\frac{\partial x_2^*}{\partial I}}_{\mathcal{E}(x_2^*, I)} = 1$$

Income and Price Elasticities

- From $p_1 x_1^*(\vec{p}, I) + p_2 x_2^*(\vec{p}, I) = I$, we have:

1. Average Income Effect = 1

$$k_1 \mathcal{E}(x_1^*, I) + k_2 \mathcal{E}(x_2^*, I) = 1$$

   – By differentiating with respect to I

   – Differentiating with respect to p_i , we have:

2. Demand for all goods has negative average response to a price increase of one good

Income and Price Elasticities

- From $p_1 x_1^*(\vec{p}, I) + p_2 x_2^*(\vec{p}, I) = I$,
  - Differentiating with respect to p_i , we have:

$$\Rightarrow x_i^*(\vec{p}, I) + p_1 \frac{\partial x_1^*}{\partial p_1} + p_2 \frac{\partial x_2^*}{\partial p_1} = 0$$

$$\Rightarrow \underbrace{\left[\frac{p_i x_i^*}{I}\right]}_{k_i} + \underbrace{\left[\frac{p_1 x_1^*}{I}\right]}_{k_1} \underbrace{\frac{p_i}{x_1^*}\frac{\partial x_1^*}{\partial p_i}}_{\mathcal{E}(x_1^*, p_i)} + \underbrace{\left[\frac{p_2 x_2^*}{I}\right]}_{k_2} \underbrace{\frac{p_i}{x_2^*}\frac{\partial x_2^*}{\partial p_i}}_{\mathcal{E}(x_2^*, p_i)} = 0$$

- Demand for all goods has negative average response to a price increase of one good

$$\Rightarrow k_1 \mathcal{E}(x_1^*, p_i) + k_2 \mathcal{E}(x_2^*, p_i) = -k_i < 0$$

Income and Price Elasticities

- From $p_1 x_1^*(\vec{p}, I) + p_2 x_2^*(\vec{p}, I) = I$, we have:

1. Average IE = 1 $k_1 \mathcal{E}(x_1^*, I) + k_2 \mathcal{E}(x_2^*, I) = 1$
2. Average demand response is negative to price increase of one good

$$k_1 \mathcal{E}(x_1^*, p_i) + k_2 \mathcal{E}(x_2^*, p_i) = -k_i < 0$$

   – Using $x_i^*(r\vec{p}, rI) = x_i^*(\vec{p}, I)$, $r > 0$,

- Can obtain $\mathcal{E}(x_1^*, p_i) + \mathcal{E}(x_2^*, p_i) + \mathcal{E}(x_i^*, I) = 0$

3. Substitute/Complement if other is price elastic/inelastic & has income elasticity = 1

Income and Price Elasticities

- Why $x_i^*(r\vec{p}, rI) = x_i^*(\vec{p}, I)$, $r > 0$?
- Money illusion! Same budget constraint…
- Differentiate with respect to r

$$\Rightarrow p_1 \frac{\partial x_i^*}{\partial p_1} + p_2 \frac{\partial x_i^*}{\partial p_2} + I \frac{\partial x_i^*}{\partial I} = 0$$

Income and Price Elasticities

- Why $x_i^*(r\vec{p}, rI) = x_i^*(\vec{p}, I)$, $r > 0$?
- Money illusion! Same budget constraint…
- Differentiate with respect to r & divide by x_i^*

$$\Rightarrow \underline{\underline{\frac{p_1}{x_i^*}\frac{\partial x_i^*}{\partial p_1}}} + \underline{\underline{\frac{p_2}{x_i^*}\frac{\partial x_i^*}{\partial p_2}}} + \underline{\underline{\frac{I}{x_i^*}\frac{\partial x_i^*}{\partial I}}} = 0$$

$$\Rightarrow \mathcal{E}(x_1^*, p_i) + \mathcal{E}(x_2^*, p_i) + \mathcal{E}(x_i^*, I) = 0$$

3. Substitute/Complement if other is price elastic/inelastic & has income elasticity = 1

Income and Price Elasticities

- From $p_1 x_1^*(\vec{p}, I) + p_2 x_2^*(\vec{p}, I) = I$, we have:

1. Average IE = 1 $k_1 \mathcal{E}(x_1^*, I) + k_2 \mathcal{E}(x_2^*, I) = 1$
2. Average demand response is negative to price increase of one good
$$k_1 \mathcal{E}(x_1^*, p_i) + k_2 \mathcal{E}(x_2^*, p_i) = -k_i < 0$$
   – Using $x_i^*(r\vec{p}, rI) = x_i^*(\vec{p}, I)$, $r > 0$,
3. Substitute/Complement if other is price elastic/inelastic & has <u>income elasticity = 1</u>
$$\mathcal{E}(x_1^*, p_i) + \mathcal{E}(x_2^*, p_i) + \mathcal{E}(x_i^*, I) = 0$$

Dual Problem: Minimizing Expenditure

- Consider the least costly way to achieve $\overline{U}$

$$M(\vec{p}, \overline{U}) = \min_{\vec{x}} \left\{ \vec{p} \cdot \vec{x} | U(\vec{x}) \geq \overline{U} \right\}$$

- How can you solve this?

$$\mathfrak{L} = -\vec{p} \cdot \vec{x} + \lambda(U(\vec{x}) - \overline{U})$$

$$(FOC) \quad \frac{\partial \mathfrak{L}}{\partial x_j} = -p_j + \lambda \frac{\partial U}{\partial x_j}(\vec{x}^*) = 0, j = 1, 2$$

$$\frac{p_1}{\frac{\partial U}{\partial x_1}} = \frac{p_2}{\frac{\partial U}{\partial x_2}} = \lambda \Rightarrow \text{Solve for } \underline{\underline{\vec{x}^c(\vec{p}, \overline{U})}}$$

Compensated Demand

Dual Problem: Minimizing Expenditure

- Can view it as the "sister" (dual) problem of:

$$\max_{\vec{x}} \left\{ U(\vec{x}) \middle| \vec{x} \geq \vec{0}, \vec{p} \cdot \vec{x} \leq I \right\}$$

- Because we have:
- Lemma 2.2-3 Duality Lemma
- LNS holds, $\vec{x}^* \in \arg\max_{\vec{x}} \left\{ U(\vec{x}) \middle| \vec{x} \geq \vec{0}, \vec{p} \cdot \vec{x} \leq I \right\}$

Max U

- Then,

$$U(\vec{x}) \geq U(\vec{x}^*) \Rightarrow \vec{p} \cdot \vec{x} \geq \vec{p} \cdot \vec{x}^*$$

- So, $\vec{x}^* \in \arg\min_{\vec{x}} \left\{ \vec{p} \cdot \vec{x} \middle| \vec{x} \geq \vec{0}, U(\vec{x}) \geq U(\vec{x}^*) \right\}$

min E

Lemma 2.2-3 Duality Lemma

- LNS holds, $\vec{x}^* \in \arg\max_{\vec{x}} \left\{ U(\vec{x}) \middle| \vec{x} \geq \vec{0}, \vec{p} \cdot \vec{x} \leq I \right\}$

Max U

- Then, $U(\vec{x}) \geq U(\vec{x}^*) \Rightarrow \vec{p} \cdot \vec{x} \geq \vec{p} \cdot \vec{x}^*$

min E

- So, $\vec{x}^* \in \arg\min_{\vec{x}} \left\{ \vec{p} \cdot \vec{x} \middle| \vec{x} \geq \vec{0}, U(\vec{x}) \geq U(\vec{x}^*) \right\}$

Proof: Consider bundle $\vec{x}'$ such that $\vec{p} \cdot \vec{x}' < I$.

- $N(\vec{x}', \vec{\delta}) \subset \left\{ \vec{x} \middle| \vec{x} \geq \vec{0}, \vec{p} \cdot \vec{x} \leq I \right\}$ for some small $\vec{\delta}$
- LNS $\Rightarrow \exists\, \vec{x}'' \in N(\vec{x}', \vec{\delta})$ such that $\vec{x}'' \succ \vec{x}'$

So $\vec{p} \cdot \vec{x}' < I \Rightarrow U(\vec{x}') < U(\vec{x}^*)$ (Equivalent!)

Expenditure Function and Value Function

- For utility $\overline{U}$ and price vector $\vec{p}$, Expenditure Function is $M(\vec{p},\overline{U}) = \min_{\vec{x}}\{\vec{p}\cdot\vec{x}|U(\vec{p}) \geq \overline{U}\}$
- Claim: The Value Function (maximized utility)

$$V(\vec{p}, I) = \max_{\vec{x}}\left\{U(\vec{x})\middle|\vec{p}\cdot\vec{x} \leq I\right\}$$

- is strictly increasing over I (by LNS).
- Then, for any $\overline{U}$, there is a unique income M such that $\overline{U} = V(\vec{p}, M)$
- Inverting this, we can solve for $M(\vec{p},\overline{U})$

Claim: Value Function is Strictly Increasing

- Claim: The Value Function is strictly increasing

$$V(\vec{p}, I) = \max_{\vec{x}} \left\{ U(\vec{x}) \middle| \vec{p} \cdot \vec{x} \le I \right\}$$

- Proof: If not, there exists $I_1 < I_2$ and $\vec{x}_1^*, \vec{x}_2^*$
  - such that $U(\vec{x}_1^*) = V(\vec{p}, I_1) \ge V(\vec{p}, I_2) = U(\vec{x}_2^*)$
- LNS yields $\vec{p} \cdot \vec{x}_1^* = I_1 < I_2$, and there exists $\vec{x}\,'$
  - such that $U(\vec{x}\,') > U(\vec{x}_1^*) \ge U(\vec{x}_2^*)$
- In neighborhood $N(\vec{x}_1^*, \vec{\delta}) \subset \left\{ \vec{x} \middle| \vec{x} \ge \vec{0}, \vec{p} \cdot \vec{x} \le I_2 \right\}$
- But this means $\vec{x}\,'$ solves $V(\vec{p}, I_2)$ not $\vec{x}_2^*$. (→←)

Dual Problem: Minimizing Expenditure

- In fact, minimizing expenditure yields:

$$\frac{p_1}{\frac{\partial U}{\partial x_1}} = \frac{p_2}{\frac{\partial U}{\partial x_2}} = \lambda$$

- Maximize Utility's FOC yields:

$$\frac{\frac{\partial U}{\partial x_1}}{p_1} = \frac{\frac{\partial U}{\partial x_2}}{p_2} = \lambda$$

- This close relationship between $\vec{x}^{\,c}(\vec{p}, \overline{U})$ and $\vec{x}(\vec{p}, I)$ indicates why they are "sisters"...

Compensated Demand

$$\vec{x}^{c}(\vec{p},\overline{U}) \text{ solves } M(\vec{p},\overline{U}) = \min_{\vec{x}} \{\vec{p}\cdot\vec{x} | U(\vec{x}) \leq \overline{U}\}$$

- By Envelope Theorem:
- Effect of "Compensated" Price Change is
  - aka Substitution Effect…

$$\frac{\partial M}{\partial p_j} = \vec{x}^{c}_{j}(\vec{p},\overline{U})$$

  - How much more does Taiwan have to pay if the price of submarines increase (to maintain the same level of defense)?

Elasticity of Substitution (Compensated Demand)

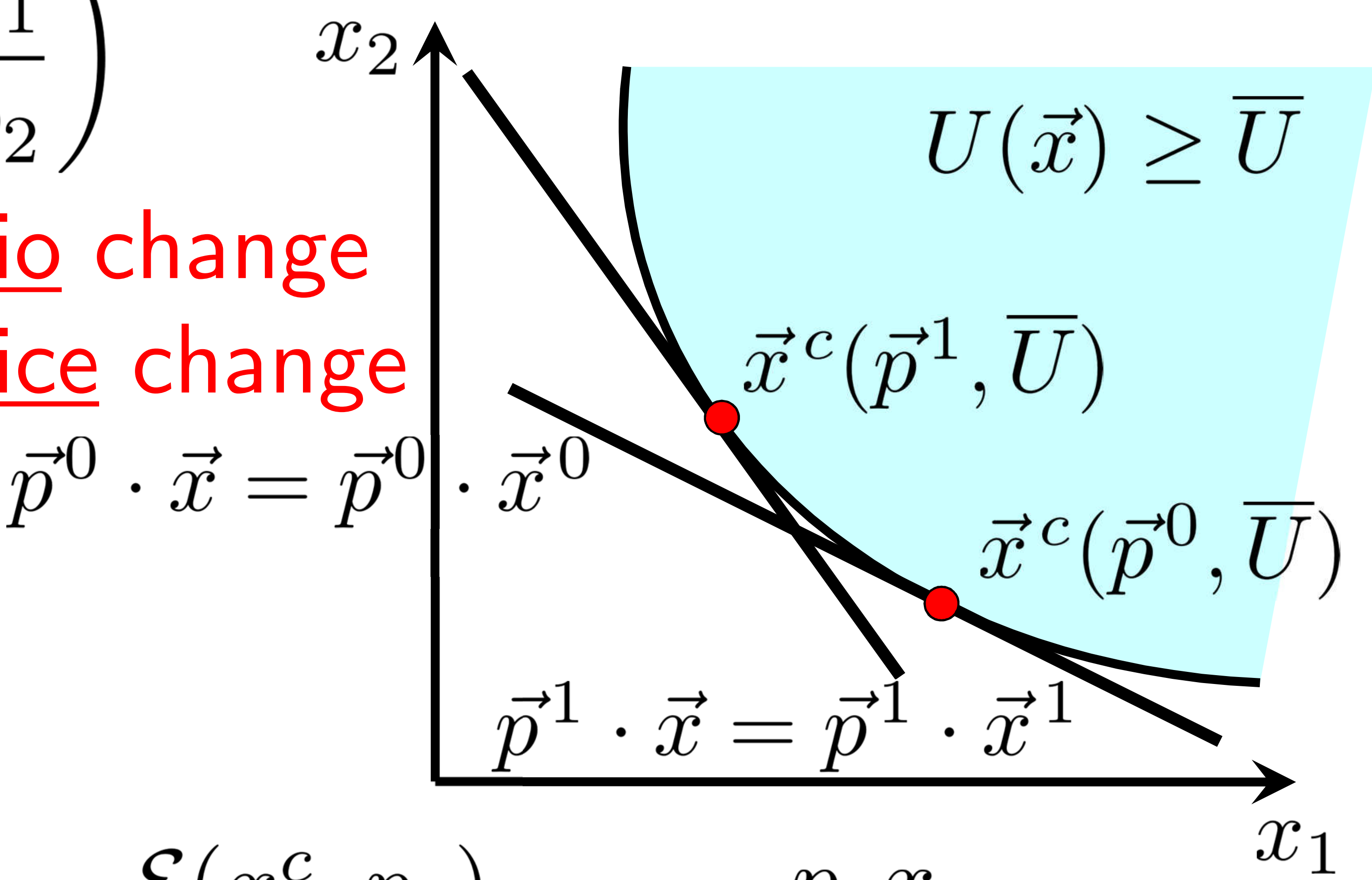
$$\sigma = \mathcal{E}\left(\frac{x_2^c}{x_1^c}, \frac{p_1}{p_2}\right)$$

Consumption ratio change in response to price change

Claim:

$$\sigma = \mathcal{E}\left(\frac{x_2^c}{x_1^c}, p_1\right)$$

$$= \frac{\mathcal{E}(x_2^c, p_1)}{k_1} = -\frac{\mathcal{E}(x_1^c, p_1)}{1-k_1},\ k_1 = \frac{p_1 x_1}{\vec{p}\cdot\vec{x}}$$



Lemma 2.2-4 $\sigma = \mathcal{E}\left(x_2^c, p_1\right) - \mathcal{E}\left(x_1^c, p_1\right)$

p.502: $\mathcal{E}(y,x) = \dfrac{x}{y} \cdot \dfrac{dy}{dx} = x\dfrac{d}{dx}\ln y = \mathcal{E}(\alpha y, \beta x)$

$$\sigma = \mathcal{E}\left(\frac{x_2^c}{x_1^c}, \frac{p_1}{p_2}\right) = \mathcal{E}\left(\frac{x_2^c}{x_1^c}, p_1\right)$$

$$= p_1\frac{d}{dp_1}\ln\left(\frac{x_2^c}{x_1^c}\right) = p_1\frac{d}{dp_1}\left(\ln x_2^c - \ln x_1^c\right)$$

$$= p_1\frac{d}{dp_1}\left(\ln x_2^c\right) - p_1\frac{d}{dp_1}\left(\ln x_1^c\right)$$

$$= \mathcal{E}\left(x_2^c, p_1\right) - \mathcal{E}\left(x_1^c, p_1\right)$$

Prop. 2.2-5
ES & Compensated Price Elasticity

- Relation between Elasticity of Substitution and Compensated Own Price Elasticity

$$1)\quad \sigma = \mathcal{E}\left(\frac{x_2^c}{x_1^c}, p_1\right) = \frac{\mathcal{E}(x_2^c, p_1)}{k_1}, \quad k_1 = \frac{p_1 x_1}{\vec{p}\cdot\vec{x}}$$

$$\left(\frac{\text{compensated cross price elasticity}}{\text{expenditure share}}\right)$$

$$2)\quad \mathcal{E}(x_1^c, p_1) = -(1-k_1)\sigma$$

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- On indifference curve, $U\left(x_1^c(p,\overline{U}), x_2^c(p,\overline{U})\right) = \overline{U}$
- Hence, $\dfrac{\partial U}{\partial x_1}\dfrac{\partial x_1^c}{\partial p_1} + \dfrac{\partial U}{\partial x_2}\dfrac{\partial x_2^c}{\partial p_1} = 0$
- By FOC, $\dfrac{p_1}{\frac{\partial U}{\partial x_1}} = \dfrac{p_2}{\frac{\partial U}{\partial x_2}} \Rightarrow \underline{\underline{p_1\dfrac{\partial x_1^c}{\partial p_1} + p_2\dfrac{\partial x_2^c}{\partial p_1} = 0}}$

$$\mathcal{E}\left(x_1^c, p_1\right) = \frac{p_1}{x_1^c}\frac{\partial x_1^c}{\partial p_1} = -\frac{p_2}{x_1^c}\frac{\partial x_2^c}{\partial p_1}$$

$$k_j = \frac{p_j x_j^c}{\vec{p}\cdot\vec{x}^c}$$

$$= -\left(\frac{p_2 x_2^c}{p_1 x_1^c}\right)\frac{p_1}{x_2^c}\frac{\partial x_2^c}{\partial p_1} = -\frac{k_2}{k_1}\mathcal{E}\left(x_2^c, p_1\right)$$

Prop. 2.2-5

ES & Compensated Price Elasticity

- Since $\mathcal{E}\left(x_1^c, p_1\right) = -\frac{k_2}{k_1}\mathcal{E}\left(x_2^c, p_1\right)$
- Lemma 2.2-4 becomes:

$$\sigma = \mathcal{E}(x_2^c, p_1) - \mathcal{E}(x_1^c, p_1)$$

$$= \mathcal{E}(x_2^c, p_1) \cdot \left(1 + \frac{k_2}{k_1}\right) = \frac{\mathcal{E}(x_2^c, p_1)}{k_1} \quad \text{...(1)}$$

$$= \mathcal{E}(x_1^c, p_1) \cdot \left(-\frac{k_1}{k_2}\right) \cdot \frac{1}{k_1} = -\frac{\mathcal{E}(x_1^c, p_1)}{k_2}$$

- Hence, $\mathcal{E}(x_1^c, p_1) = -k_2\sigma = -(1 - k_1)\sigma$ ...(2)

Compensated own price elasticity bounded/approx. by ES!

Elasticity of Substitution (Compensated Demand)

- Verify that $\sigma = \theta$ for CES:
- Since $x_1 = \left(\dfrac{\alpha_1}{\alpha_2}\dfrac{p_2}{p_1}\right)^{\theta} \cdot x_2 \;\Rightarrow \dfrac{x_2}{x_1} = \left(\dfrac{\alpha_2}{\alpha_1} \cdot \dfrac{p_1}{p_2}\right)^{\theta}$

$$\Rightarrow \ln\left(\frac{x_2^c}{x_1^c}\right) = \theta(\ln p_1 - \ln p_2 + \ln\alpha_2 - \ln\alpha_1)$$

$$\Rightarrow \sigma = \mathcal{E}\left(\frac{x_2^c}{x_1^c}, p_1\right) = p_1 \cdot \frac{\partial}{\partial p_1}\left[\ln\left(\frac{x_2^c}{x_1^c}\right)\right]$$

$$= p_1 \cdot \frac{\theta}{p_1} = \theta$$

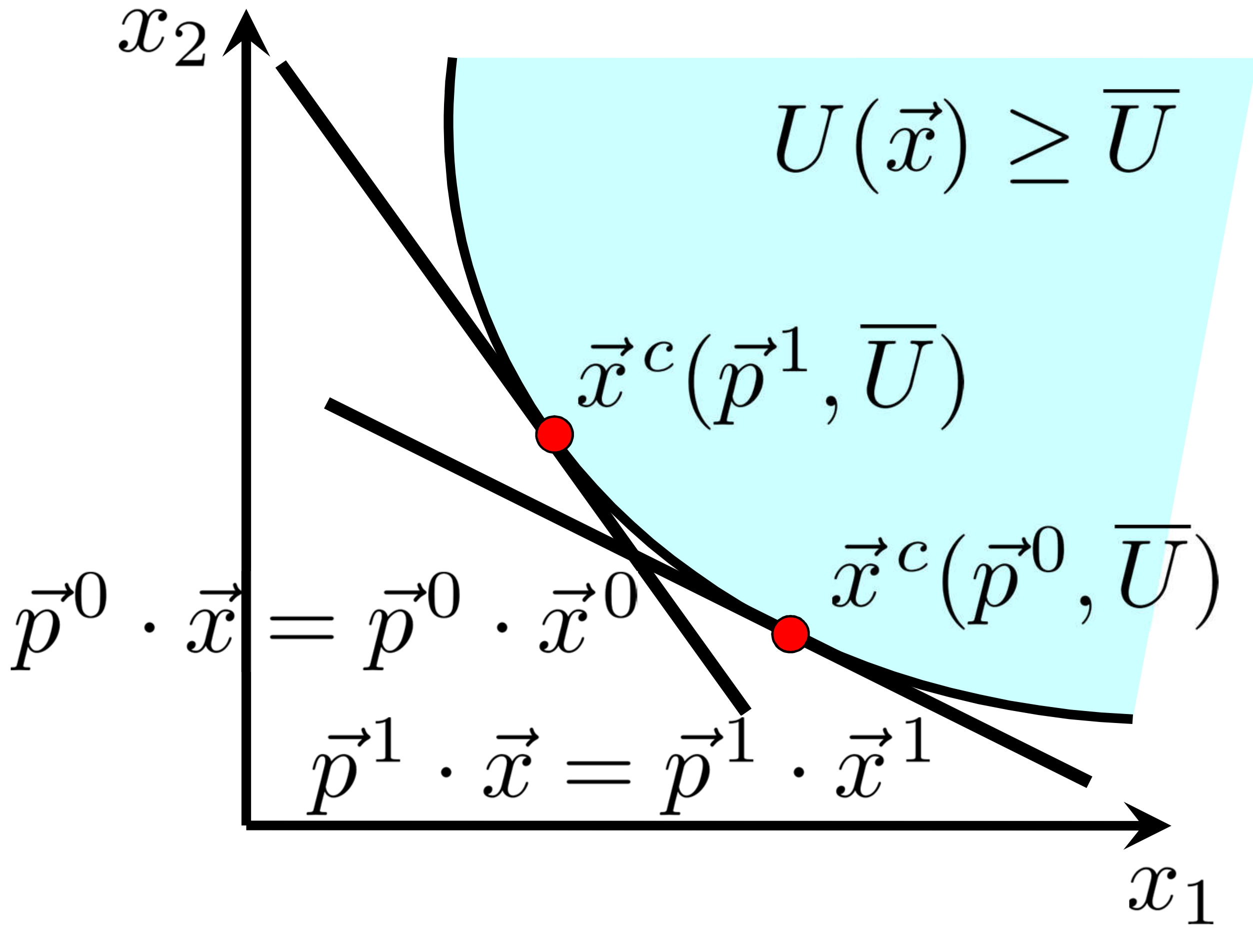
Summary for Elasticity of Substitution

- 1. $\sigma = \mathcal{E}\left(\frac{x_2^c}{x_1^c}, p_1\right)$
- 2. $= \frac{\mathcal{E}(x_2^c, p_1)}{k_1}$

$$= -\frac{\mathcal{E}(x_1^c, p_1)}{1 - k_1}$$

$$k_1 = \frac{p_1 x_1}{\vec{p} \cdot \vec{x}}$$

- 3. $\sigma = \theta$ for CES…



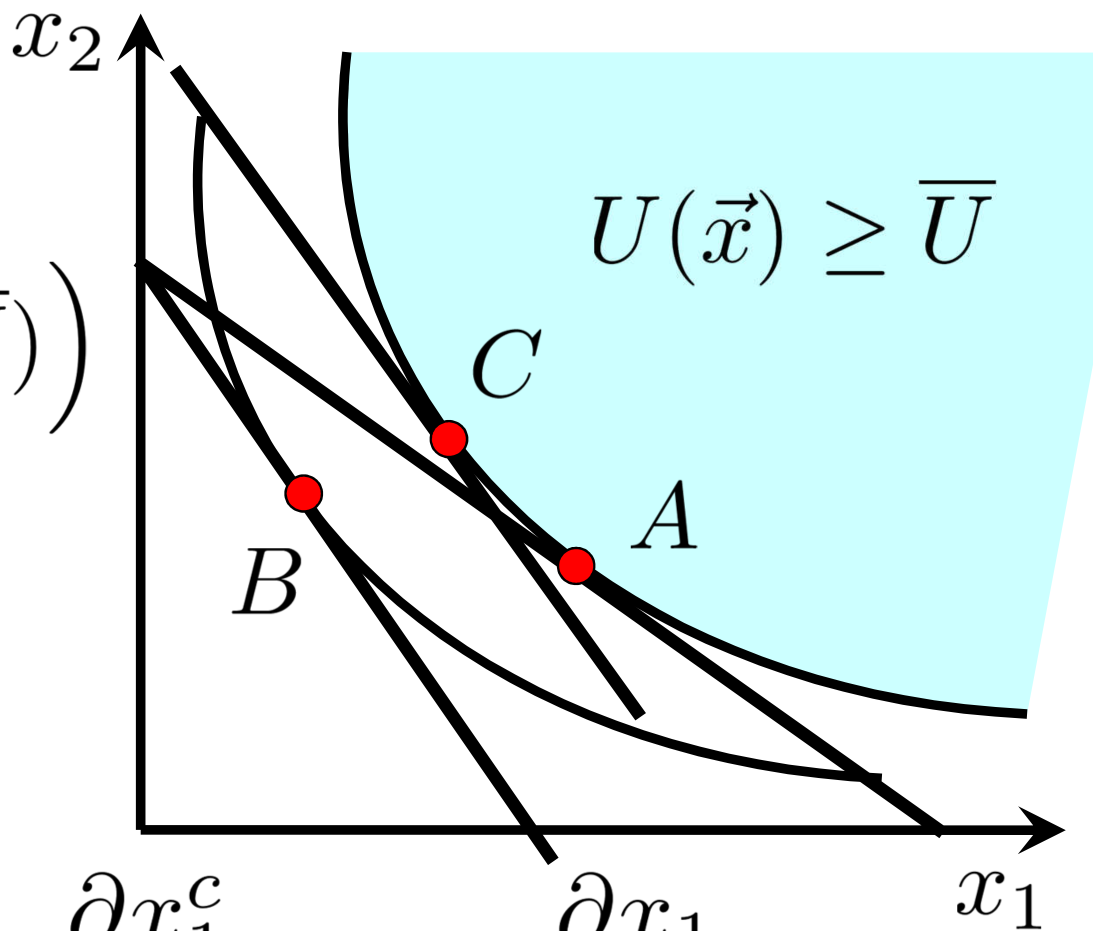
Total Price Effect
= Income Effect + Substitution Effect

- For $M(\vec{p},\overline{U})$ & $x_1(\vec{p},I)$
- Compensated Demand:

$$x_1^c(\vec{p},\overline{U}) = x_1\left(\vec{p}, M(\vec{p},\overline{U})\right)$$

$$\frac{\partial x_1^c}{\partial p_1} = \frac{\partial x_1}{\partial p_1} + \frac{\partial x_1}{\partial I}\cdot\underline{\underline{\frac{\partial M}{\partial p_1}}}$$

$$\left(\frac{\partial M}{\partial p_1} = x_1\right)$$



- Slutsky Equation:

$$\underbrace{\frac{\partial x_1}{\partial p_1}}_{A\to B} = \underbrace{\frac{\partial x_1^c}{\partial p_1}}_{A\to C} \underbrace{- x_1 \cdot \frac{\partial x_1}{\partial I}}_{C\to B}$$

Decomposition of Own Price Elasticity

- Slutsky Equation: $\dfrac{\partial x_1}{\partial p_1} = \dfrac{\partial x_1^c}{\partial p_1} - x_1 \cdot \dfrac{\partial x_1}{\partial I}$
- Elasticity Version:

$$\frac{p_1}{x_1}\frac{\partial x_1}{\partial p_1} = \frac{p_1}{x_1}\frac{\partial x_1^c}{\partial p_1} - \frac{p_1 x_1}{I}\frac{I}{x_1}\cdot\frac{\partial x_1}{\partial I}$$

- Or, $\underline{\mathcal{E}(x_1, p_1)} = \underline{\underline{\mathcal{E}(x_1^c, p_1)}} - \underline{\underline{k_1 \cdot \mathcal{E}(x_1, I)}}$

$$= \underline{\underline{-(1-k_1)\sigma}} - \underline{\underline{k_1 \cdot \mathcal{E}(x_1, I)}}$$

Substitution Effect　　Income Effect

– Own price elasticity = weighted average of elasticity of substitution and income elasticity

Summary of 2.2

- Consumer Problem: Maximize Utility
- Income Effect
- Dual Problem: Minimize Expenditure
- Substitution Effect:
  - =Compensated Price Effect
  - Elasticity of Substitution
- Total Price Effect:
  - = Compensated Price Effect + Income Effect
- Homework: Exercise 2.2-4 (Optional: 2.2-5)

In-Class Homework: Exercise 2.2-2

- Show that the price effect on compensated demand is

$$\frac{\partial M}{\partial p_j}(\vec{p}, \overline{U}) = x_j^c(\vec{p}, \overline{U})$$

  - Hint: Convert expenditure minimization into a maximization problem, write down the Lagrangian and use the Envelope Theorem…

In-Class Homework: Exercise 2.2-3

- [Elasticity of Substitution]

a) Show that $\mathcal{E}(y(x), z(x)) = \dfrac{\frac{d}{dx}\ln y}{\frac{d}{dx}\ln z}$.

b) Use this to show that $\mathcal{E}\left(\dfrac{1}{y}, \dfrac{1}{x}\right) = \mathcal{E}(y, x)$

and that $\mathcal{E}\left(\dfrac{y_2}{y_1}, x\right) = \mathcal{E}(y_2, x) - \mathcal{E}(y_1, x)$

c) Use these results to prove Lemma 2.2-4.

In-Class Homework: Exercise 2.2-6

- [Parallel Income Expansion Paths]
- A consumer faces price vector p, has income I and utility function $U(\vec{x}) = -\alpha_1 e^{-Ax_1} - \alpha_2 e^{-Ax_2}$

a) Show that her optimal consumption bundle satisfies the following: $x_2 - x_1 = a + b \ln \frac{p_1}{p_2}$

b) Depict her Income Expansion Paths.