A.5 Sufficient Conditions for a Maximum (pp. 513–520)

Def: f is concave on $[a,b]$ if

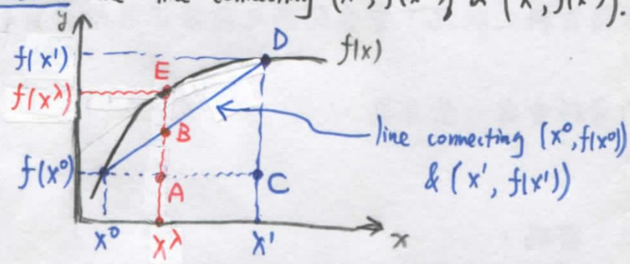
For $x^0, x' \in [a,b]$, $x^\lambda = (1-\lambda)x^0 + \lambda x'$, $0<\lambda<1$

$$f(x^\lambda) \geq (1-\lambda)\underline{f(x^0)} + \lambda \underline{f(x')}$$

f is strictly concave if

$$f(x^\lambda) > (1-\lambda)f(x^0) + \lambda f(x').$$

Note: This means that function f between x^0 & x' lies above the line connecting $(x^0, f(x^0))$ & $(x', f(x'))$.



Why? $x^\lambda - x^0 = \lambda(x' - x^0) \Rightarrow \overline{AB} = \lambda \cdot \overline{CD}$. (linear) (A.5-1)

For f concave, we have

$$\underbrace{f(x^\lambda) - f(x^0)}_{\overline{AE}} \geq \lambda \cdot \underbrace{[f(x') - f(x^0)]}_{\overline{CD}} = \overline{AB}. \quad \text{(A.5-2)} \quad \#$$

Prop. A.5-1: If f is concave & diff. at x^0 & x', then

$$\frac{df}{dx}(x^0)\cdot(x'-x^0) \geq f(x') - f(x^0) \geq \frac{df}{dx}(x')\cdot(x'-x^0)$$

(pf) Eq. (A.5-1) & (A.5-2) imply:

$$f(x^\lambda) - f(x^0) \geq \left[\frac{x^\lambda - x^0}{x'-x^0}\right]\cdot[f(x') - f(x^0)]$$

$$\Rightarrow \underbrace{\frac{f(x^\lambda) - f(x^0)}{x^\lambda - x^0}}_{\xrightarrow{\lambda \to 0}\ \frac{df}{dx}(x^0)} \cdot (x'-x^0) \geq f(x') - f(x^0)$$

Similarly, $x' - x^\lambda = (1-\lambda)(x'-x^0)$. Rearranging the def. of concavity, $f(x') - f(x^\lambda) \leq (1-\lambda)\cdot[f(x') - f(x^0)]$

$$\Rightarrow f(x') - f(x^\lambda) \leq \left(\frac{x'-x^\lambda}{x'-x^0}\right)\cdot[f(x') - f(x^0)]$$

$$\Rightarrow f(x') - f(x^0) \geq \underbrace{\frac{f(x') - f(x^\lambda)}{x' - x^\lambda}}_{\xrightarrow{\lambda \to 1}\ \frac{df}{dx}(x')} \cdot (x' - x^0) \quad \#$$

Corollary A.5-2: If f is strictly concave on interval X & diff. at $x^0 \in X$, then $\frac{df}{dx}(x^0)\cdot(x'-x^0) > f(x') - f(x^0)$

Moreover, if f is also diff. at $x' > x^0$, then $\frac{df}{dx}(x^0) > \frac{df}{dx}(x')$.

(pf) By Prop. A.5-1, $\frac{df}{dx}(x^0)\cdot(x^\lambda - x^0) \geq f(x^\lambda) - f(x^0)$

f strictly concave $\Rightarrow f(x^\lambda) - f(x^0) > \lambda\cdot[f(x') - f(x^0)]$.

Hence, $\frac{df}{dx}(x^0)\cdot(x^\lambda - x^0) > \lambda\cdot[f(x') - f(x^0)]$

$$= \left(\frac{x^\lambda - x^0}{x'-x^0}\right)\cdot[f(x') - f(x^0)]$$

$$\therefore \frac{df}{dx}(x^0)\cdot(x'-x^0) > f(x') - f(x^0).$$

Moreover, if f is diff. at $x' > x^0$, by Prop. A.5-1,

$$\frac{df}{dx}(x^0)\cdot(x'-x^0) > f(x') - f(x^0) \geq \frac{df}{dx}(x')\cdot(x'-x^0). \quad \#$$

Note: This means that all tangent lines lie above the graph of f.

In fact, this is "iff":

Prop. A.5-3: A diff. function f is concave on interval X iff for $x^0, x' \in X$, $f(x') \leq f(x^0) + \frac{df}{dx}(x^0)\cdot(x' - x^0)$. (A.5-3)

(pf) ($\Rightarrow$) By Prop. A.5-2.

($\Leftarrow$) For $x^0, x' \in X$, $x^\lambda = (1-\lambda)x^0 + \lambda\cdot x'$, by (A.5-3):

$$\begin{cases} f(x^0) \leq f(x^\lambda) + \frac{df}{dx}(x^\lambda)\cdot(x^0 - x^\lambda) \cdots\cdots ① \\ f(x') \leq f(x^\lambda) + \frac{df}{dx}(x^\lambda)\cdot(x' - x^\lambda) \cdots\cdots ② \end{cases}$$

$①(1-\lambda) + ②\cdot\lambda$: $(1-\lambda)f(x^0) + \lambda\cdot f(x') \leq f(x^\lambda) + 0$. #

Note: This also means the $\frac{df}{dx} \downarrow$ as $x \uparrow$, or $\frac{d^2f}{dx^2} \leq 0$.

In fact, this is also "iff":

Prop. A.5-4: A twice diff. function f is concave on the interval X iff for $x^0, x' \in X$, $\frac{d^2f}{dx^2}(x) \leq 0 \quad \forall x \in X$.

(pf) ($\Rightarrow$) By Prop. A.5-2.

($\Leftarrow$) Consider $g(\lambda) = f(x^0 + \lambda(x'-x^0))$, by chain rule,

$$\frac{dg}{d\lambda}(\lambda) = (x'-x^0)\cdot\frac{df}{dx}(x^\lambda), \quad \frac{d^2g}{d\lambda^2}(\lambda) = (x'-x^0)^2\cdot\frac{d^2f}{dx^2}(x^\lambda).$$

Since $\frac{d^2f}{dx^2} \leq 0 \ \forall x \in X$, $\frac{d^2g}{d\lambda^2} \leq 0$ & $\frac{dg}{d\lambda}$ decreasing on $[0,1]$

$$\therefore \frac{dg}{d\lambda}(0) = \underline{(x'-x^0)\cdot\frac{df}{dx}(x^0)} = \int_0^1 \frac{dg}{d\lambda}(0)\, d\lambda$$

$$\leq \int_0^1 \frac{dg}{d\lambda}(\lambda)\cdot d\lambda = g(1) - g(0) = \underline{f(x') - f(x^0)}$$

So by Prop. A.5-3, f is concave. #

Prop. A.5-5 (Sufficient Conditions for a Maximum)

Suppose f is concave on $[a,b]$ & for some $x^0 \in (a,b)$, $\frac{df}{dx}(x^0) = 0$.

Then, $f(x) \leq f(x^0) \ \forall x \in [a,b]$.

(pf) By Prop. A.5-3, $\forall x \in [a,b]$, $f(x) \leq f(x^0) + \frac{df}{dx}(x^0)\cdot(x - x^0) = f(x^0)$

since $\frac{df}{dx}(x^0) = 0$. #

Note: FOC is enough (if f is concave).

A.5 Sufficient Conditions for a Maximum (Continued)

Concavity might be too strong. Can relax to quasi-concavity.

Def: (Local Property) f satisfies property P locally at x° if property P holds in some neighborhood of x°.

EX: (Local Maximum) By Corollary A.5-2, if $\frac{df}{dx}(x^\circ)=0$ (FOC holds at x°) & f is locally strictly concave at x°, then $f(x^\circ) > f(x')$ $\forall x'$ in a neighborhood of x°.
i.e. f has a local maximum at x°.

Def: f is quasi-concave over $[a,b]$ if
for $x^\circ, x' \in [a,b]$, $x^\lambda = (1-\lambda)x^\circ + \lambda \cdot x'$, $0<\lambda<1$,
$f(x^\lambda) \geq \min\{f(x^\circ), f(x')\}$.

Note: Quasi-concavity implies that a local maximum is also a global maximum.

Prop. A.5-6 (Quasi-concavity & Sufficient Conditions for a Max.)
If $\frac{df}{dx}(x^\circ)=0$ & f is locally strictly concave at $x^\circ \in [a,b]$, then f has a local maximum at x°.
If f is quasi-concave on $[a,b]$, then $f(x) < f(x^\circ)$ $\forall x \in [a,b]$.

(pf) EX. gives the first statement: $\exists$ neighborhood $N(x^\circ, \delta)$ s.t.
$f(x) < f(x^\circ)$ $\quad \forall x \in x^\circ$.
Consider $[a,b]$, if the exists $x' \in [a,b]$ s.t. $f(x') > f(x^\circ)$, then if f is quasi-concave, $f(x^\lambda) \geqslant \min\{f(x'), f(x^\circ)\} = f(x^\circ)$ for all $x^\lambda = (1-\lambda)x^\circ + \lambda x'$, $0<\lambda<1$. But this means that $\exists\, x^\lambda \in N(x^\circ, \delta)$ s.t. $f(x^\lambda) \geqslant f(x^\circ)$, contradiction. #

By setting $g=-f$, we get the flip-side conditions for minimum:

Def: f is convex on $[a,b]$ if for $x^\circ, x' \in [a,b]$, $x^\lambda = (1-\lambda)x^\circ + \lambda \cdot x'$, $0<\lambda<1$, we have
$f(x^\lambda) \underline{\underline{\leq}} (1-\lambda) f(x^\circ) + \lambda \cdot f(x')$
(i.e. $g=-f$ is concave!)
f is strictly convex if
$f(x^\lambda) < (1-\lambda) f(x^\circ) + \lambda \cdot f(x')$

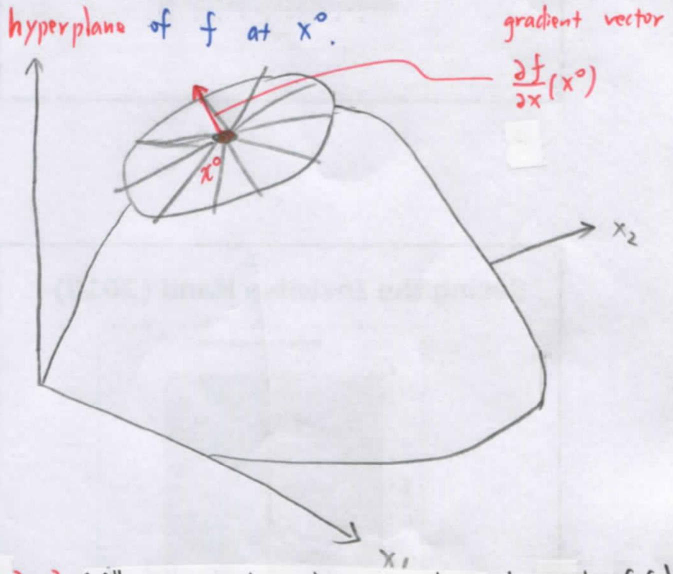
Prop. A.5-7 (Sufficient Conditions for a Minimum)
Suppose f is convex on $D_f = \mathbb{R}$ & for some x°, $\frac{df}{dx}(x^\circ)=0$.
Then, $f(x) \geq f(x^\circ)$ $\forall x \in \mathbb{R}$.

(pf) Set $g=-f$ & use Prop. A.5-5. #

B.2 Concave Functions of Vectors (pp. 532-

Def: f is concave on convex set $X \subseteq \mathbb{R}^n$ if
for $x^\circ, x' \in X$, $x^\lambda = (1-\lambda)x^\circ + \lambda x'$, $0<\lambda<1$
$f(x^\lambda) \geq (1-\lambda) f(x^\circ) + \lambda f(x')$
(x°, x', x^λ are all vectors in $\mathbb{R}^n$)
f is strictly concave if
$f(x^\lambda) > (1-\lambda) f(x^\circ) + \lambda f(x')$

Def: $H = \{x \mid \underbrace{\frac{\partial f}{\partial x}(x^\circ)}_{\text{(gradient)}} \cdot (x - x^\circ) = 0\}$ is the tangent hyperplane of f at x°.



Prop. B.2-3 (All tangent hyperplanes lie above the graph of f)
A diff. function f is concave on convex set $X \subseteq \mathbb{R}^n$ iff
for $x^\circ, x' \in X$, $f(x') \leq f(x^\circ) + \frac{\partial f}{\partial x}(x^\circ) \cdot (x' - x^\circ)$.

(pf) ($\Leftarrow$) [Copy Prop. A.5-3]
$f(x^\circ) \leq f(x^\lambda) + \frac{\partial f}{\partial x}(x^\lambda) \cdot (x^\lambda - x^\circ)$ ---- ①
$f(x') \leq f(x^\lambda) \cdot \frac{\partial f}{\partial x}(x^\lambda) \cdot (x^\lambda - x^\circ)$ ---- ②
①$\cdot(1-\lambda)$ + ②$\cdot\lambda$: $(1-\lambda) f(x^\circ) + \lambda f(x') \leq f(x^\lambda) + 0$. *

($\Rightarrow$) Consider $g(\lambda) = f(x^\lambda) = f(x^\circ + \lambda(x' - x^\circ)) \geq (1-\lambda) f(x^\circ) + \lambda f(x')$ (f concave)
$g'(\lambda) = \frac{\partial f}{\partial x}(x^\lambda) \cdot (x' - x^\circ)$
Since $g(0) = f(x^\circ)$, $g(\lambda) - g(0) \geqslant \lambda \cdot [f(x') - f(x^\circ)]$
Or, $\frac{g(\lambda) - g(0)}{\lambda - 0} \geqslant f(x') - f(x^\circ)$
$\xrightarrow{(\lambda \to 0)}$ $g'(0) = \frac{\partial f}{\partial x}(x^\circ) \cdot (x' - x^\circ)$. #

Prop. B.2-4
A twice diff. function f is concave on the convex set $X \subseteq \mathbb{R}^n$ iff for $x^\circ, x \in X$, the quadratic form $q(x)$ is negative semi-definite.

$$q(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(x^\circ)(x_i - x_i^\circ)(x_j - x_j^\circ)$$

EX: In $\mathbb{R}^2$, this requires $\frac{\partial^2 f}{\partial x_j^2} \leq 0$, $j=1,2$ & $\frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 f}{\partial x_2^2} - \left(\frac{\partial^2 f}{\partial x_1 \partial x_2}\right)^2 \geqslant 0$

(pf of Prop. B.2-4)

For x^0, $x' = x^0 + z$, define $x^\theta = x^0 + \theta z$ & $g(\theta) = f(x^0 + \theta z)$.

($\Rightarrow$) f is concave $\Rightarrow g(\theta)$ is concave.

(pf) Consider θ^0, θ' & $\theta^\lambda = (1-\lambda)\theta^0 + \lambda\theta'$,

$$g(\theta^\lambda) = f(x^0 + (1-\lambda)\theta^0 z + \lambda\theta' z) = f\big((1-\lambda)(x^0+\theta^0 z) + \lambda(x^0+\theta' z)\big)$$

$$\geq (1-\lambda) f(x^0+\theta^0 z) + \lambda f(x^0+\theta' z) \qquad (\because f \text{ is concave})$$

$$= (1-\lambda) g(\theta^0) + \lambda g(\theta') \ \#$$

g is concave $\Rightarrow g''(0) \leq 0$

But $g'(\theta) = \frac{\partial f}{\partial x}(x^0+\theta z)\cdot z = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x^0+\theta z)\cdot z_i$

$$\Rightarrow g''(\theta) = \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(x^0+\theta z) z_i z_j$$

Hence, $g''(0) = \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(x^0)\cdot(x_i - x_i^0)\cdot(x_j - x_j^0) \leq 0$. #

($\Leftarrow$) $g(x) \leq 0 \Rightarrow g''(\theta) \leq 0 \Rightarrow g'(\theta)$ decreasing on $[0,1]$

$\Rightarrow g'(\theta) \leq g'(0) \ \forall \theta \in [0,1]$

$$\therefore f(x') - f(x^0) = g(1) - g(0) = \int_0^1 g'(\theta)d\theta \leq \int_0^1 g'(0)d\theta$$

$$= g'(0) = \frac{\partial f}{\partial x}(x^0)\cdot(x' - x^0).$$

Hence, by Prop. B.2-3, f is concave. #

Exercise: Show that the Cobb-Douglas function $f(x) = x_1^\alpha x_2^\beta$, $x \geq 0$ $(\alpha, \beta \geq 0)$ is concave if $\alpha + \beta \leq 1$.

Prop. B.2-5 (Sum of Concave Functions)

f, g concave $\Rightarrow (f+g)$ is concave. EX: $f(x) = \sum_j \ln x_j$ is concave

Prop. B.2-6 (Concave Function of a Function)

$h(x) = g(f(x))$ is concave if g is concave and either

(a) g is increasing & f is concave, or (b) f is linear.

EX: $f(x) = (x_1^\alpha + x_2^\beta)^\gamma$, $\alpha, \beta, \gamma \in (0,1]$, $x \gg 0$ is concave.

Def: f is quasi-concave on convex set X if

$x^0, x' \in X$, $x^\lambda = (1-\lambda)x^0 + \lambda x'$, $0<\lambda<1$,

$f(x^\lambda) \geq \min\{f(x^0), f(x')\}$.

f is quasi-convex if $f(x^\lambda) \leq \min\{f(x^0), f(x')\}$.

Prop. B.2-7 f is concave $\Rightarrow f$ is quasi-concave.

Prop. B.2-8 f is concave, g is increasing $\Rightarrow h(x) = g(f(x))$ is quasi-concave.

EX: $f(x) = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, $x \geq 0$ is quasi-concave!

Def: f is homogeneous of degree 1 if $f(\mu x) = \mu f(x) \ \forall x \in \mathbb{R}^n_+, \mu > 0$

Prop. B.2-9 f is quasi-concave & homogeneous of degree 1 $\Rightarrow f$ concave.

(pf) For x^0, x', find θ s.t. $\underbrace{f((1-\lambda)x^0)}_{=(1-\lambda)f(x^0)} = \theta \underbrace{f(\lambda x')}_{=\lambda f(x')}$

$\Rightarrow (1-\lambda)f(x^0) + \lambda f(x') = (1+\theta) f(\lambda x')$

Consider $\frac{\theta}{1+\theta}(1-\lambda)x^0 + \frac{1}{1+\theta}\theta\lambda x' = \left(\frac{\theta}{1+\theta}\right)\cdot\left[(1-\lambda)x^0 + \lambda x'\right]$,

f is quasi-concave $\Rightarrow$

$$f\left(\frac{\theta}{1+\theta}\cdot((1-\lambda)x^0 + \lambda x')\right) \geq \min\{f((1-\lambda)x^0), f(\theta\lambda x')\} = f(\theta\lambda x')$$

$$\Rightarrow \frac{\theta}{1+\theta} f((1-\lambda)x^0 + \lambda x') \geq \theta f(\lambda x')$$

$$\Rightarrow f((1-\lambda)x^0 + \lambda x') \geq (1+\theta) f(\lambda x') = (1-\lambda)f(x^0) + \lambda f(x'). \ \#$$

EX: $f(x) = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, $x \geq 0$, $\alpha_1 + \cdots + \alpha_n = 1 \Rightarrow$ homogeneous of degree 1.

$\Rightarrow f$ is concave (since f is quasi-concave).

Def: $C(x^0) = \{x \mid f(x) = f(x^0)\}$ is the contour set of f.

$C_U(x^0) = \{x \mid f(x) \geq f(x^0)\}$ is the upper contour set.

Prop. B.2-11 f is quasi-concave iff the upper contour set of f is convex.

(pf) ($\Rightarrow$) f is quasi-concave. For any $\hat{x}$ & $x^0, x' \in C_U(\hat{x})$,

$\Rightarrow f(x^0) \geq f(\hat{x})$, $f(x') \geq f(\hat{x})$. for $x^\lambda = (1-\lambda)x^0 + \lambda x'$,

$f(x^\lambda) \geq \min\{f(x^0), f(x')\} \geq f(\hat{x})$,

i.e. $x^\lambda \in C_U(\hat{x})$. #

($\Leftarrow$) For any x^0, x', Without loss of generality, assume $f(x') \geq f(x^0)$

$\therefore x' \in C_U(x^0)$. By convex upper contour set,

$x^\lambda = (1-\lambda)x^0 + \lambda x' \in C_U(x')$

$\Rightarrow f(x^\lambda) \geq f(x^0) = \min\{f(x^0), f(x')\} \Rightarrow f$ quasi-concave #

Prop. B.2-12 $f \in C^1$ is quasi-concave iff

$$f(x) \geq f(x^0) \Rightarrow \frac{\partial f}{\partial x}(x^0)\cdot(x - x^0) \geq 0$$