Envelope Theorem

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(Lecture 3, Micro Theory I)
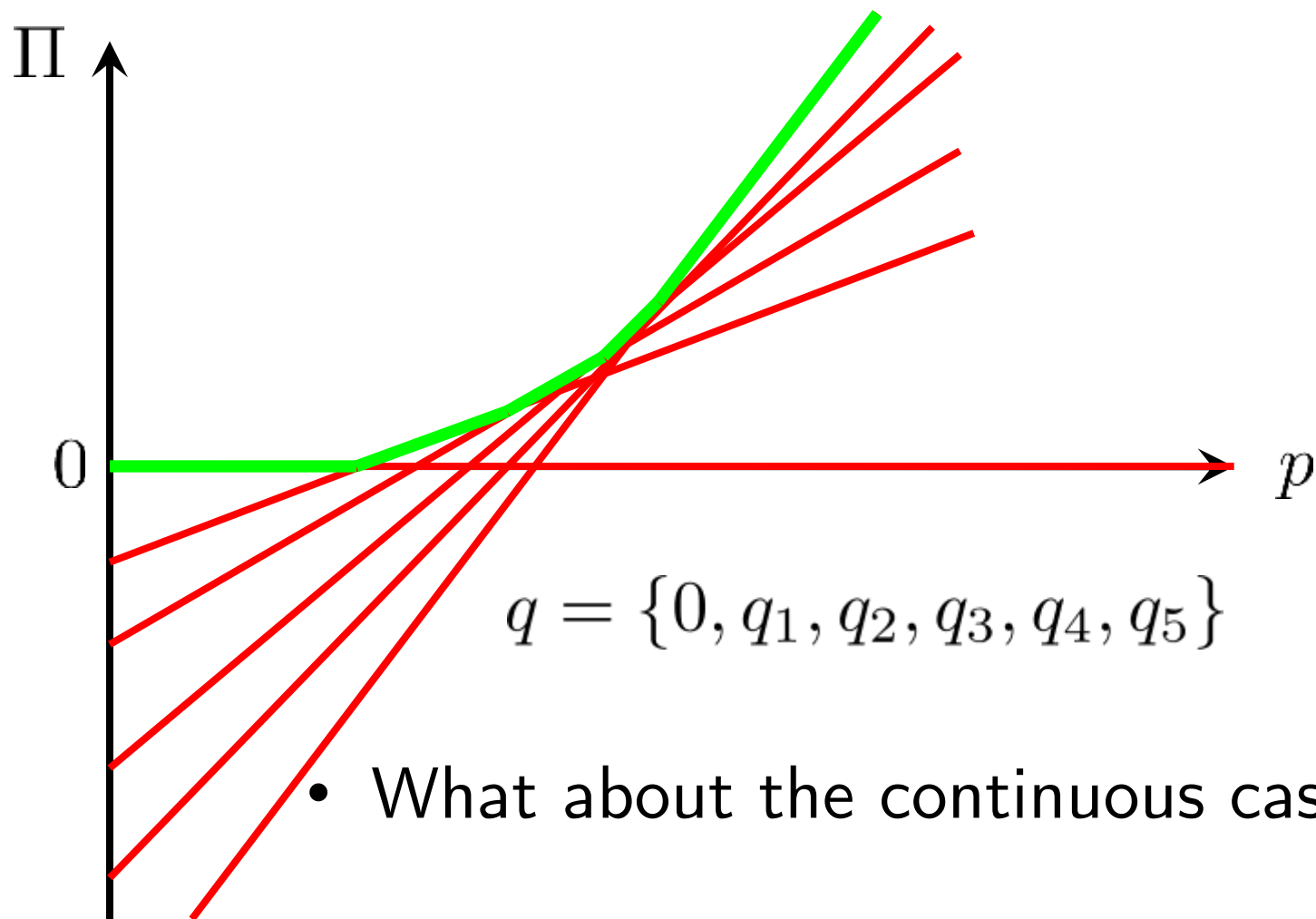
- A price-taking firm has cost $C(q)$
  - Can sell as much as it wishes at fix price p
- Profit is $\pi = p \cdot q - C(q)$
- Given a change in prices p, how would profit change (as the firm re-optimizes output q)?
  - Direct Effect: $\Delta p \cdot q$
  - Indirect Effect: $\Delta \pi$ due to $q \to q'$
- First assume only three possible outputs...
$$q \in Q = \{0, q_1, q_2\}$$
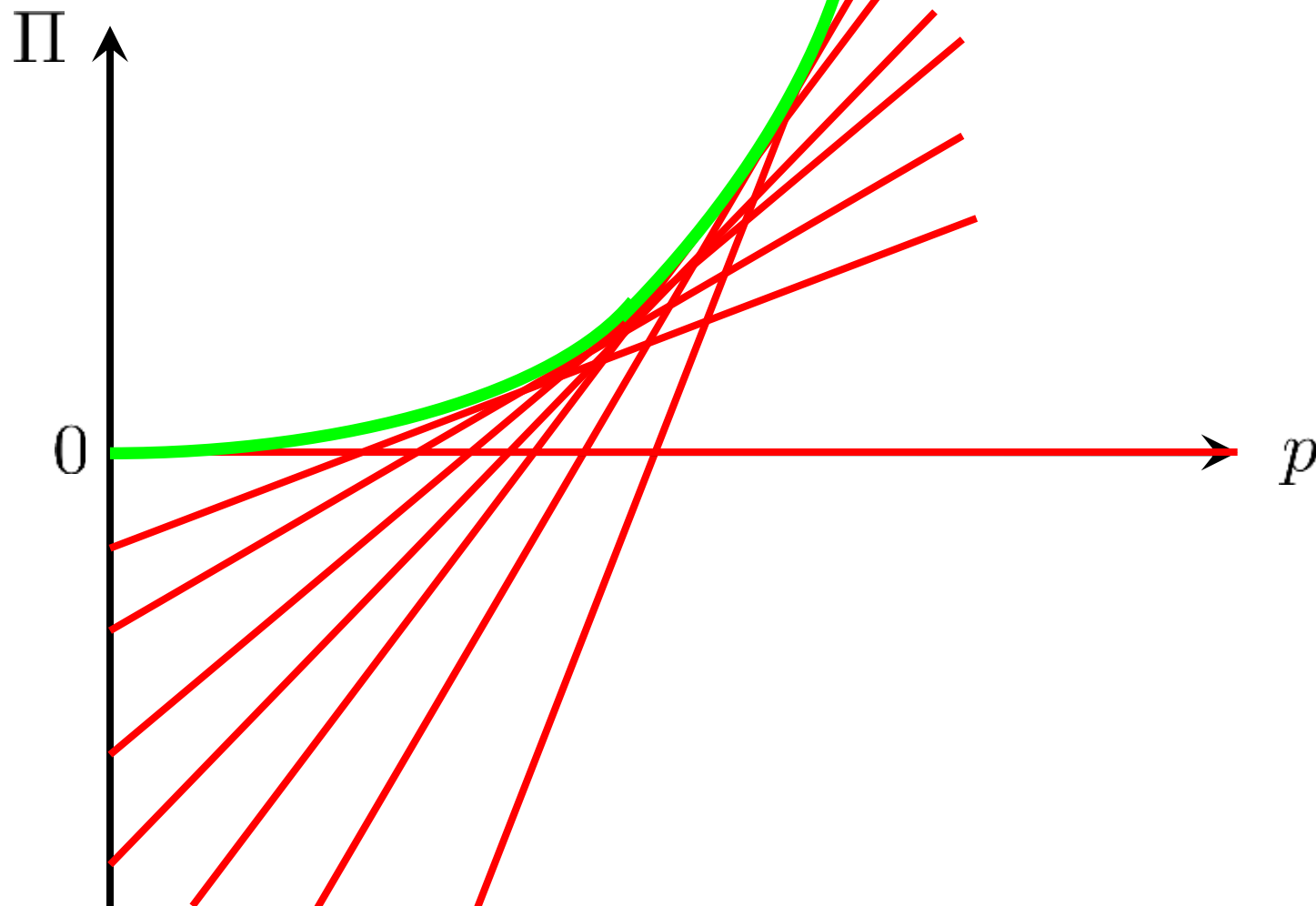 - Profit is straight line for each possible output
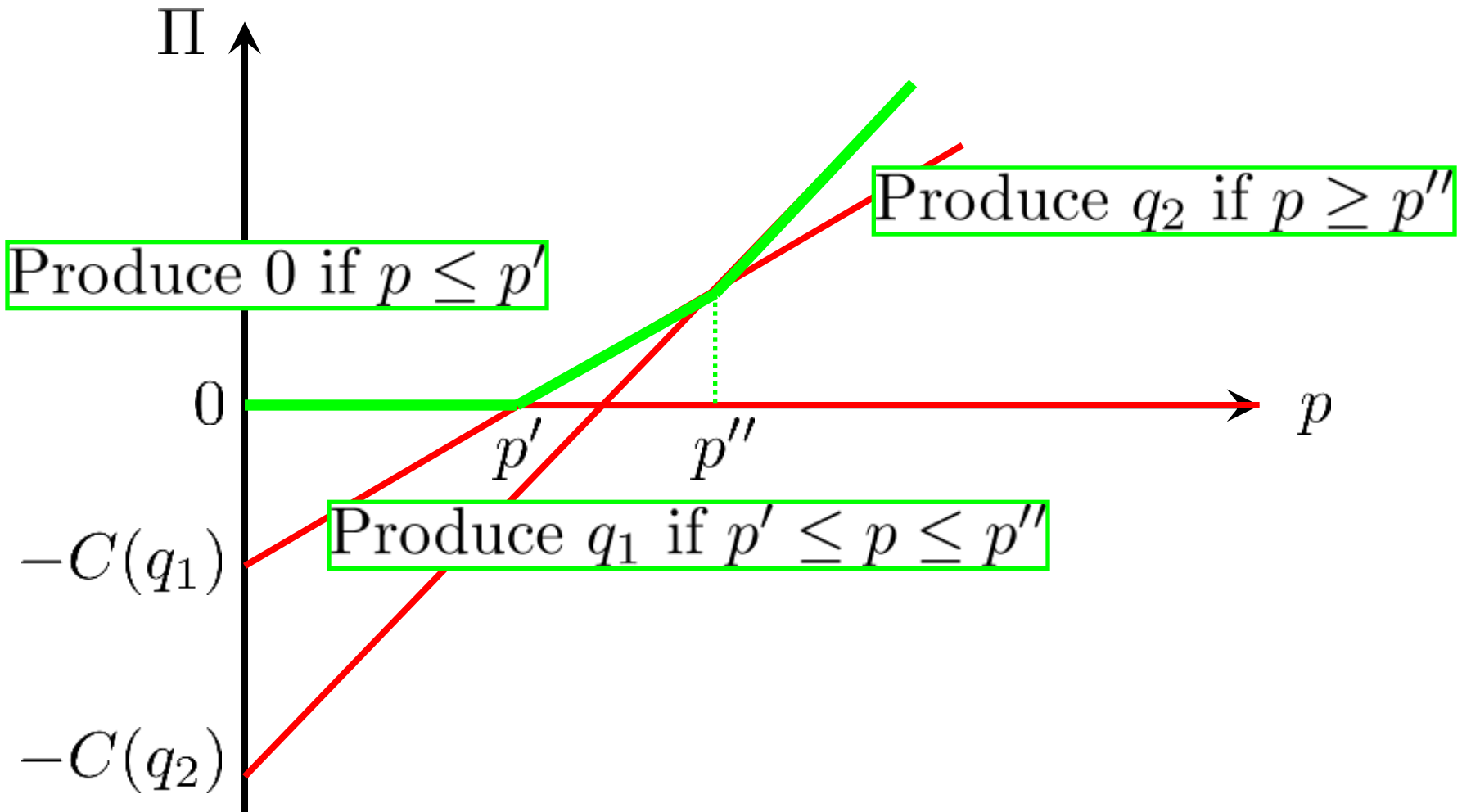
$$\Pi_2 = -C(q_2) + p \cdot q_2$$

$$\Pi_1 = -C(q_1) + p \cdot q_1$$

$$\Pi_0 = -C(0) = 0$$

$$q \in Q = \{0, q_1, q_2\}$$

Π

Produce 0 if $p \leq p'$

Produce q_2 if $p \geq p''$

0

p' p'' p

$-C(q_1)$

Produce q_1 if $p' \leq p \leq p''$

$-C(q_2)$

$$q = \{0, q_1, q_2, q_3, q_4, q_5\}$$

- What about the continuous case?
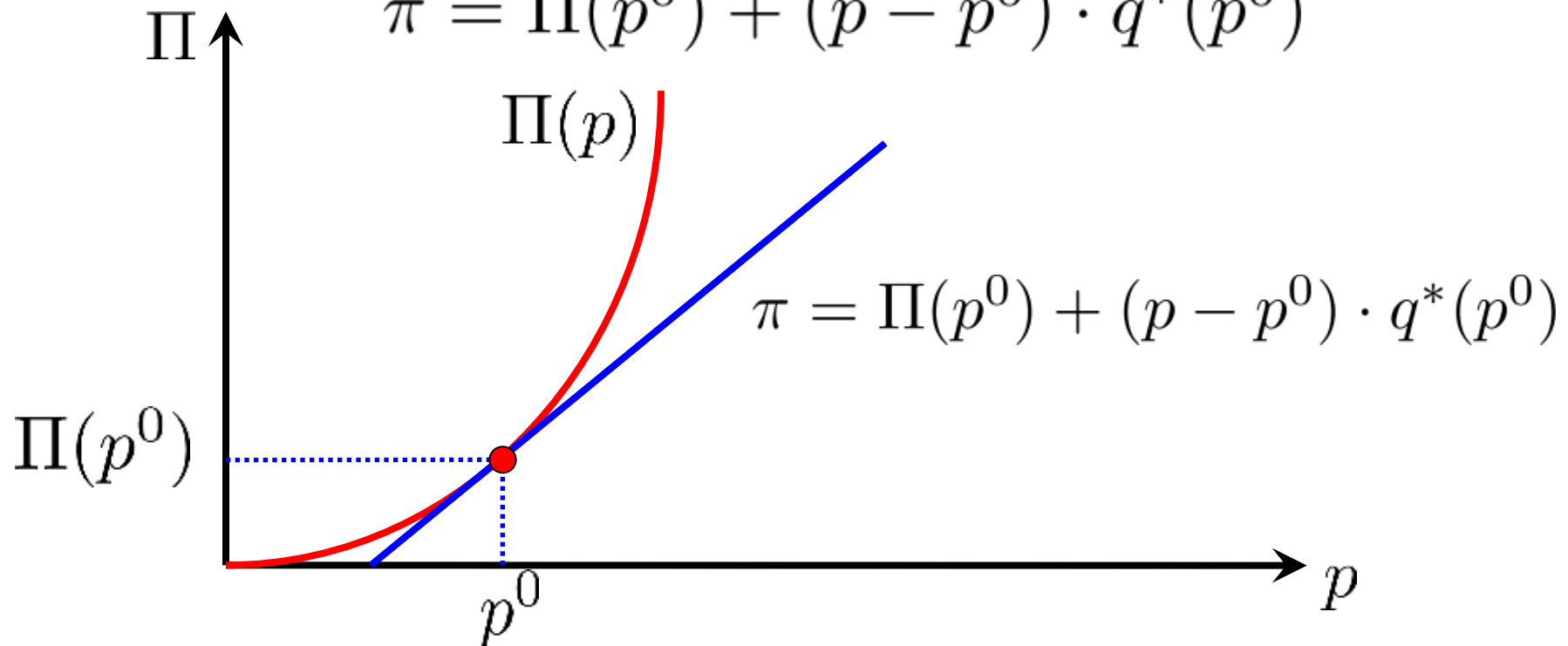
- Output can be any real number
- Firm solves $q^*(p)$ to $\max \{\pi = p \cdot q - C(q)\}$
- Maximized profit is $\Pi(p) = p \cdot q^*(p) - C(q^*(p))$
- Initial output price p^0 (fixed)
  - Initial output $q^*(p^0)$
  - Initial profit $\Pi(p^0)$
- Profit (with fixed output) is
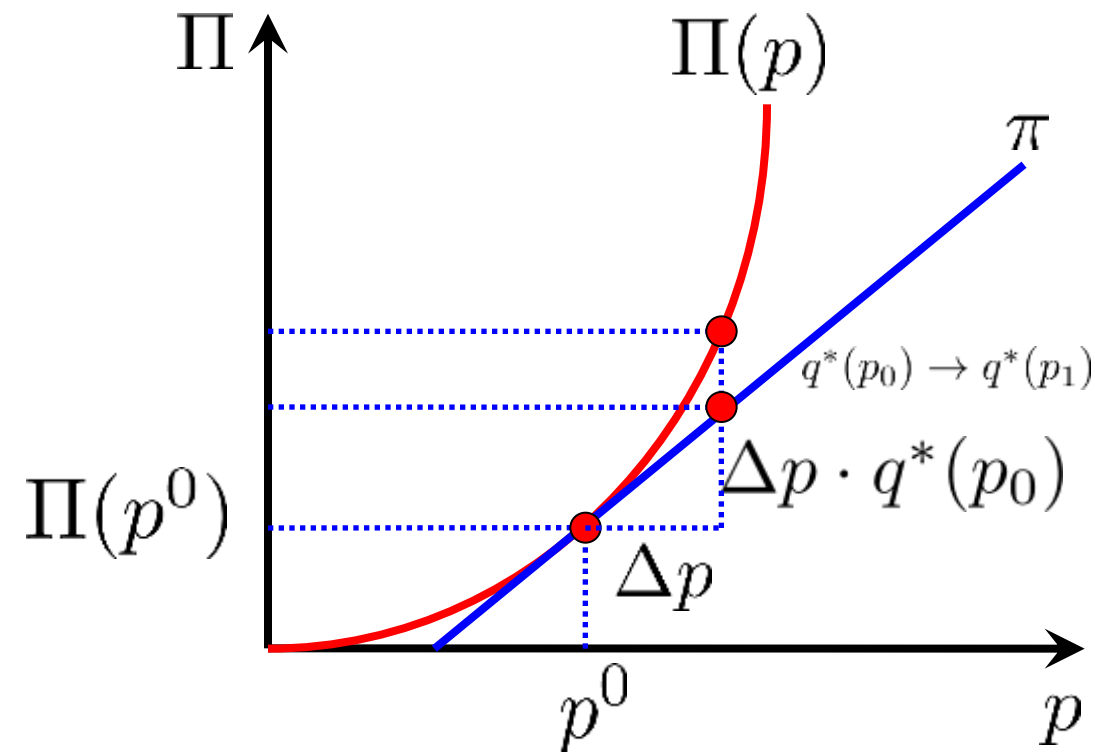$$\pi = \Pi(p^0) + (p - p^0) \cdot q^*(p^0)$$

- Fixing output, increase in price changes profit by $q^*(p)$ per dollar, so (fixed output) profit is

$$\pi = \Pi(p^0) + (p - p^0) \cdot q^*(p^0)$$

$$\pi = \Pi(p^0) + (p - p^0) \cdot q^*(p^0)$$

$$\frac{\partial \Pi}{\partial p}(p^0) = q^*(p^0)$$

$\Pi(p)$

π

$q^*(p_0) \rightarrow q^*(p_1)$

$\Delta p \cdot q^*(p_0)$
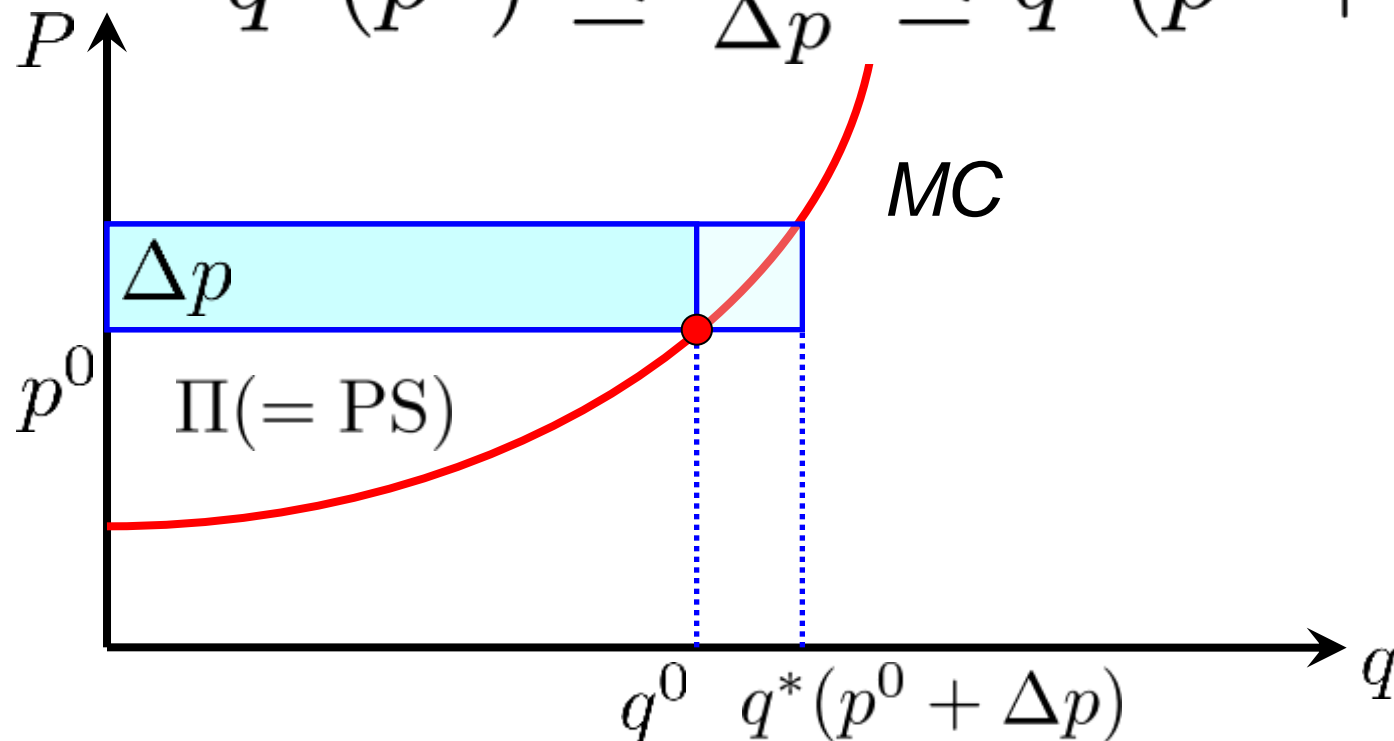
$\Pi(p^0)$

Δp

p^0

p

- Firm cannot be worse off if it can change quantity
- $\Pi(p)$ is above π
  - Tangent to π if $\Pi(p)$ smooth
- Total effect = Direct effect only
  - Ignore indirect eff.

$$q^*(p^0)\Delta p \leq \Delta \Pi \leq q^*(p^0 + \Delta p)\Delta p$$

$$q^*(p^0) \leq \frac{\Delta \Pi}{\Delta p} \leq q^*(p^0 + \Delta p)$$

- Assume:
  - X is closed and bounded, (Feasible output)
  - f is continuously differentiable (Profit)

$$x(\alpha) = \arg\max_{x \in X} \{f(x, \alpha)\} \quad \text{is unique,} \boxed{q^*(p^0)}$$

$\boxed{\alpha = p}$

- Then, value function, $F(\alpha) = \max_{x \in X} \{f(x, \alpha)\}$

$\boxed{\Pi(p)}$

is differentiable and $\dfrac{dF}{d\alpha} = \dfrac{\partial f}{\partial \alpha}(x(\alpha), \alpha)$.

(Only Direct Effect)

Proposition 1.3-3: Envelope Theorem I

- Direct Effect = Total Effect (at the margin)
- This only allows the maximand to be affected by the parameter change...
- To allow for both the maximand and the constraints to be affected by the parameter change, need slightly stronger assumptions...

- For $F(\alpha) = \max\limits_{x} \{f(x, \alpha) | h(x, \alpha) \geq 0, x \geq 0\}$

$$\mathcal{L}(x, \lambda, \alpha) = f(x, \alpha) + \lambda h(x, \alpha)$$

- Suppose:
  - f and h are continuously differentiable
  - $x(\alpha), \lambda(\alpha)$ unique solutions; CQ hold.
  - $x(\alpha)$ and $\lambda(\alpha)$ continuously differentiable
  - at α^0 (implicit function theorem applies)

- Then, $\dfrac{\partial F}{\partial \alpha}(\alpha^0) = \dfrac{\partial \mathcal{L}}{\partial \alpha}\left(x(\alpha^0), \lambda(\alpha^0), \alpha\right)$.

- Hunghai is a price-taking firm making jpads
  - Sell 3,000 jpads to Pineapple at price $p_i = \$100$
  - Total Cost is $C(q) = \$180,000$
- What is the elasticity of profit w.r.t. price
  - If output is held fixed?

    $$\epsilon(\Pi, p_i) = \frac{p_i}{\Pi} \frac{\partial \Pi}{\partial p_i}$$
  - If Hung-Hai responds optimally to price change?
- Hunghai sees a new opportunity and sells 1,500 Vii's to Rentientang at price $p_w = \$200$
  - Production of jpads drop to 2,400, total cost rises to $300,000. Can you calculate the new $\epsilon(\Pi, p_i)$?

- Hunghai is a price-taking firm making jpads
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$$
\begin{aligned}
\Pi &= p \times q - C(q) \\
&= \$100 \times 3,000 - \$180,000 = \$120,000
\end{aligned}
$$

$$
\frac{\partial \Pi}{\partial p_i} = q_i = 3,000 \Rightarrow \epsilon = \frac{p_i}{\Pi}\frac{\partial \Pi}{\partial p_i} = \frac{\$100 \cdot 3,000}{\$120,000} = \frac{5}{2}
$$

- Hunghai's elasticity of profit wrt. jpad price is 2.5 for both fixed and variable output (by ET!)

- Hunghai sees a new opportunity and sells 1,500 Vii's to Rentientang at price $p_w = \$200$
  - Production of jpads drop to 2,400, price $p_i = \$100$
  - total cost rises to $300,000. Calculate new $\epsilon(\Pi, p_i)$

$$\Pi = \$100 \times 2,400 + \$200 \times 1,500 - \$300,000$$

$$= \$240,000$$

$$\frac{\partial \Pi}{\partial p_i} = q_i' = 2,400 \Rightarrow \epsilon = \frac{p_i}{\Pi}\frac{\partial \Pi}{\partial p_i} = \frac{\$100 \cdot 2,400}{\$240,000} = 1$$

What does this all mean?

- Hunghai used to only produce jpads

- Since it is a price-taker, if Pineapple Corp. decides to lower prices by 10%, Hunghai's profit would decrease by 25%

- Even if Hunghai tries to re-optimize! (ET)

- After diversifying to producing also Vii's, it's profit is now less prone to Pineapple's price cuts (lowers by 10% if prices are cut by 10%)

- Isn't this what firms in Hsinchu Science Park do?

Summary of 1.3

- Re-maximize under environmental change
  - Direct Effect: Change in profit (objective function)
  - Indirect Effect: Change due to re-optimization
- Envelope Theorem(s):
  - Only have Direct Effect at the margin

- Homework: Exercise 1.3-4 (Optional: 1.3-3)