Supporting Prices and Convexity

Joseph Tao-yi Wang

2013/9/11

(Lecture 1, Micro Theory I)

Overview of Chapter 1

- Theory of Constrained Maximization
  - Why should we care about this?
- What is Economics?
- Economics is the study of how society manages its scarce resources (Mankiw, Ch.1)
  - "Economics is the science which studies human behavior as a relationship between given ends and scarce means which have alternative uses." (Lionel Robbins, 1932)

Overview of Chapter 1

- Other Historical Accounts:
  - Economics is the "study of how societies use scarce resources to produce valuable commodities and distribute them among different people." (Paul A. Samuelson, 1948)
- My View: Economics is a study of institutions and human behavior (reactions to institutions)
- Either way, constrained maximization is key...

Tools Introduced in Chapter 1

1. Supporting Hyperplanes (and Convexity)
2. First Order Conditions (Kuhn-Tucker)
3. Envelope Theorem

- But why do I need to know the math?
- When does Coase conjecture work?
  - It depends—Math makes these predictions precise
- What happens if you ignore the conditions required for theory to work? (Recall 2008/09!)

Publication Reward Problem

- Example: How should NTU reward its professors to publish journal articles?
  - Should NTU pay, say, NT$300,000 per article published in Science or Nature?
- Well, it depends...
- Peek the answer ahead:
  - Yes, if the production set is convex.
  - No, if, for example, there is initial increasing returns to scale.

Supporting Prices

- More generally, can prices and profit maximization provide appropriate incentives for all efficient production plans?
  - Is there a price vector that supports each efficient production plan?
- (Yes, but when?)
- Need some definitions first...

Production Plant

- A production plant can:
- produce n outputs $q = (q_1, \cdots, q_n)$
- using up to m inputs $z = (z_1, \cdots, z_m)$
- Production Plan (z, q)
- Production Set $Y \subset \mathbf{R}_+^{m+n}$

  =Set of all Feasible Production Plan
- Production Vector (treat inputs as negative)

  $y = (-z, q) = (-z_1, \cdots, -z_m, q_1, \cdots, q_n)$

Production Set and Profits

- Production vector
  $y = (y_1, \cdots, y_{m+n}) = (-z_1, \cdots, -z_m, q_1, \cdots, q_n)$
- Production Set $\mathcal{Y} \subset \mathbf{R}^{m+n}$

  =Set of Feasible Production Plan
- Price vector $p = (p_1, \cdots, p_{m+n})$
- Profit $\Pi = \underbrace{\sum_{i=m+1}^{m+n} p_i y_i}_{\text{total revenue}} - \underbrace{\sum_{i=1}^{m} p_i z_i}_{\text{total cost}} = p \cdot y$

EX: Production Function and Production Set

- A professor has 25 units of "brain-power"
- Allocates z_1 units to produce TSSCI papers
- Produce $q_1 = 4\sqrt{z_1}$ (Production Function)
- Production Set

$$Y_1 = \left\{ (z_1, q_1) \middle| z_1 \geq 0, q_1 \leq 4\sqrt{z_1} \right\}$$

- Treating inputs as negatives, $y = (-z, q)$
- Production Set is

$$\mathcal{Y}_1 = \left\{ (y_1, y_2) \middle| -16y_1 - y_2^2 \geq 0 \right\}$$

Production Efficiency

- A production plan y is wasteful if another plan in $\mathcal{Y}$ achieves larger output with smaller input
- $\overline{y}$ is production efficient (=non-wasteful) if

  There is no $y \in \mathcal{Y}$ such that $y > \overline{y}$

  – Note: $y \geq \overline{y}$ if $y_j \geq \overline{y}_j$ for all *j*

  – $y > \overline{y}$ if inequality is strict for some *j*

  – $y \gg \overline{y}$ if inequality is strict for all *j*

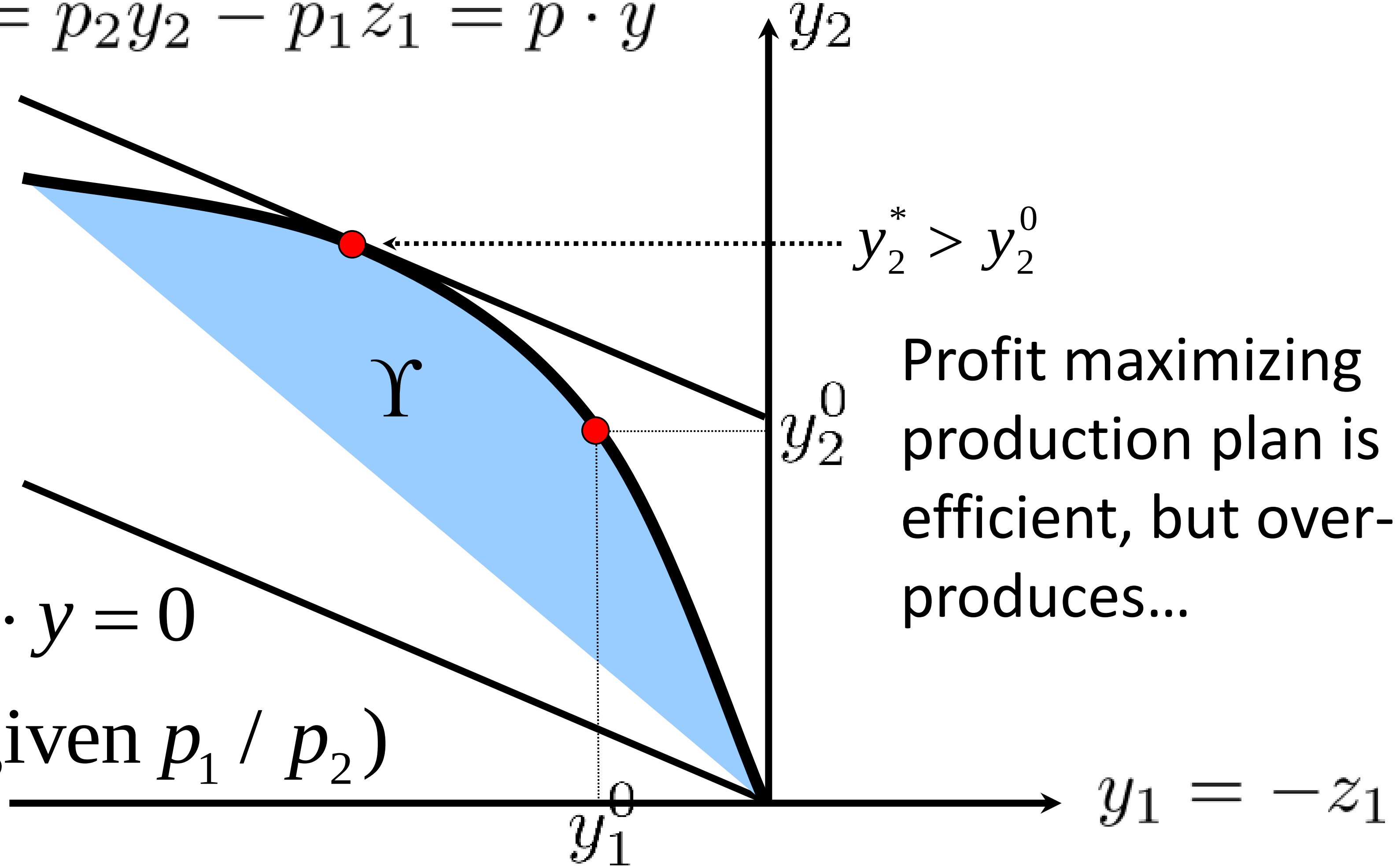
EX: Can Prices Support Efficient Production?

- A professor has 25 units of "brain-power"
- Allocates y_1 units to produce TSSCI papers
- Price of brain-power is p_1
- Production Set $\mathcal{Y}_1$

- Can we induce production target y_2^0?
- With piece-rate prize p_2?

Can Prices Support Efficient Production Plan?

$\Pi = p_2 y_2 - p_1 z_1 = p \cdot y$



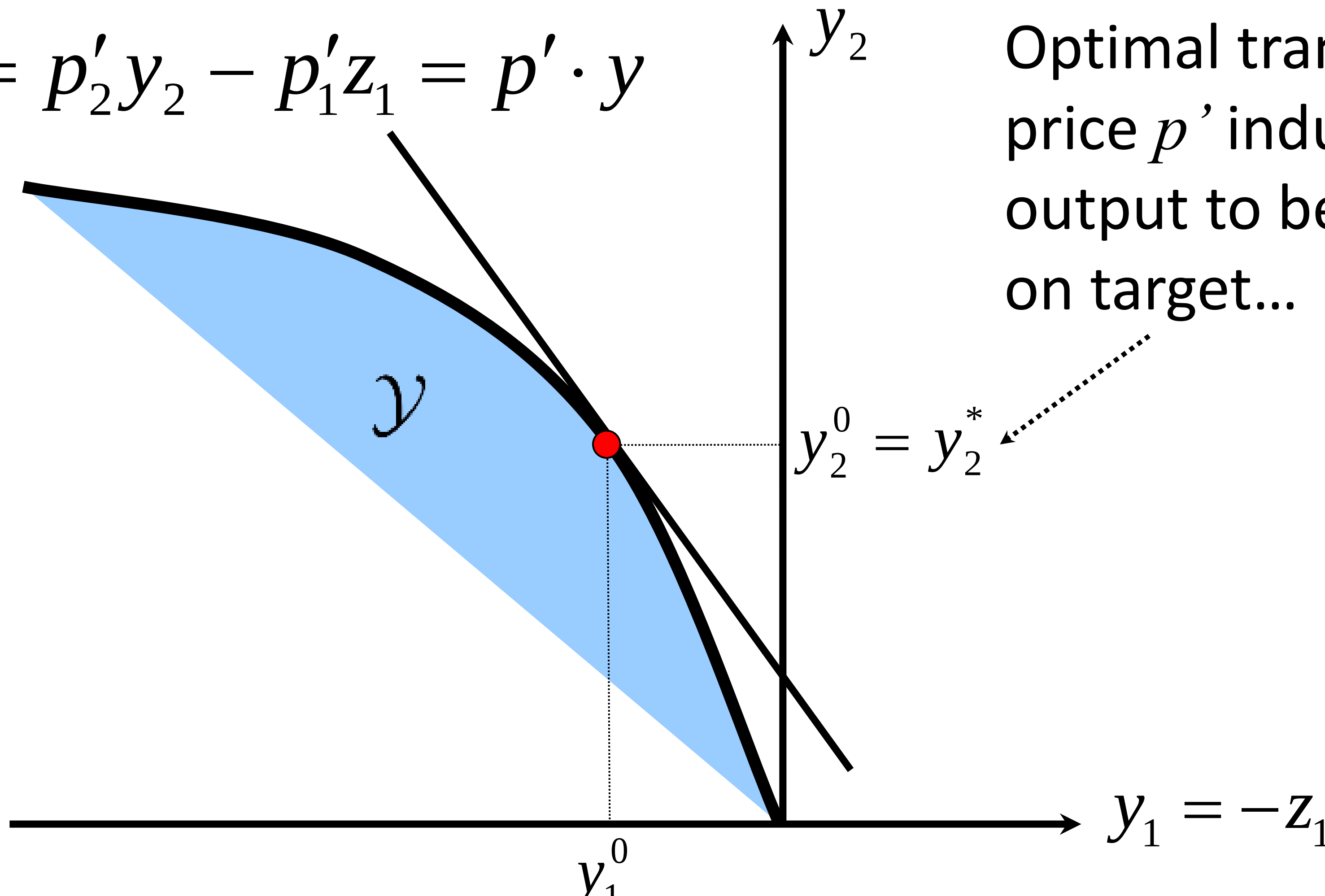
$p \cdot y = 0$

(given p_1 / p_2)

Profit maximizing production plan is efficient, but over-produces...

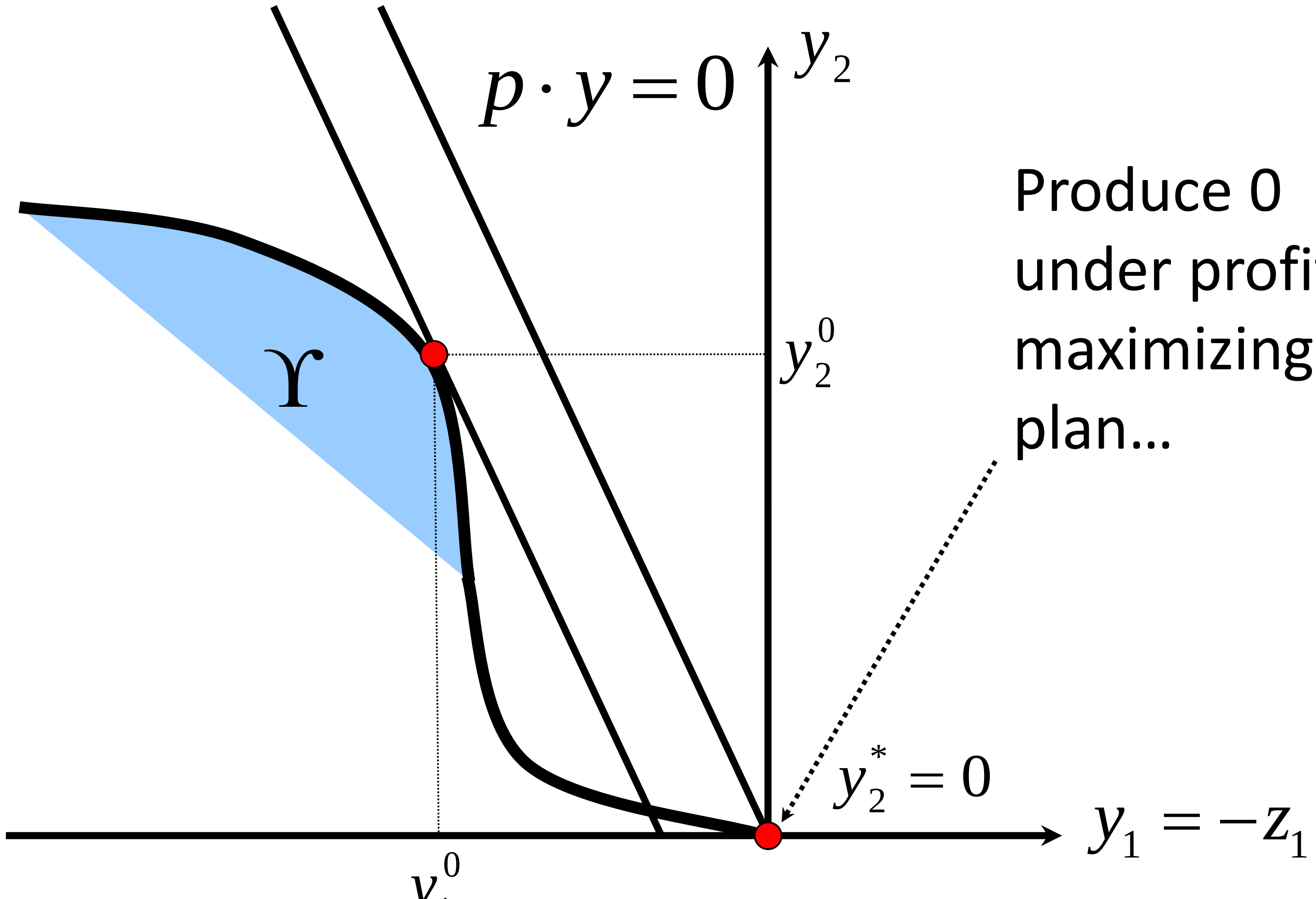
Too High? Let's Lower the Transfer Price...

$$\Pi = p_2' y_2 - p_1' z_1 = p' \cdot y$$



Optimal transfer price p' induces output to be right on target...

Will this Always Work?



Produce 0 under profit maximizing plan…

What Made It Fail?

- The last production set was NOT convex.
- Production Set $\mathcal{Y}_1$ is convex if for any y^0, y^1
- Its convex combination (for $0 < \lambda < 1$)

$$y^\lambda = (1-\lambda)y^0 + \lambda y^1 \in \mathcal{Y}_1$$

- (is also in the production set)
- Is it true that we can use prices to guide production decisions as long as production sets are convex?

Supporting Hyperplane Theorem

Proposition 1.1-1 (Supporting Hyperplane)

- Suppose $\mathcal{Y} \subset \mathbf{R}^n$ is non-empty and convex,
- And y^0 lies on the boundary of $\mathcal{Y}$
- Then, there exists $p \neq 0$ such that

i. For all $y \in \mathcal{Y}$, $p \cdot y \leq p \cdot y^0$

- Proof: For the general case, see Appendix C.

Special Case of Supporting Hyperplane Thm

- Consider special case where
- Production set $\mathcal{Y}$ is the upper contour set

$$\mathcal{Y} = \{y | g(y) \geq g(y^0)\}, \quad g \text{ is differentiable}$$

- Suppose the gradient vector is non-zero at y^0
- The linear approximation of g at y^0 is:

$$\overline{g}(y) = g(y^0) + \frac{\partial g}{\partial y}(y^0) \cdot (y - y^0)$$

- If $\mathcal{Y}$ is convex, it lies in upper contour set of $\overline{g}$

Special Case of Supporting Hyperplane Thm

- **Lemma 1.1-2**
- If g is differentiable and $\mathcal{Y} = \{y | g(y) \geq g(y^0)\}$ is convex, then

$$y \in \mathcal{Y} \quad \Rightarrow \quad \frac{\partial g}{\partial y}(y^0) \cdot (y - y^0) \geq 0$$

$$g(y) \geq g(y^0)$$

- This tells us how to calculate the supporting prices (under this special case):
- For boundary point y^0, choose $p = -\frac{\partial g}{\partial y}(y^0)$

From Lemma to Supporting Hyperplane Thm

- If g is differentiable and $\mathcal{Y} = \{y | g(y) \geq g(y^0)\}$ is convex, then (by lemma)

$$\begin{aligned} y \in \mathcal{Y} \ &\Rightarrow \ -p \cdot (y - y^0) \geq 0 \\ &\Rightarrow \ p \cdot y \leq p \cdot y^0 \end{aligned}$$

- This gives us part (i) of S. H. T.

Supporting Hyperplane Theorem

Proposition 1.1-1 (Supporting Hyperplane)

- Suppose $\mathcal{Y} \subset \mathbf{R}^n$ is non-empty and convex,
- And y^0 lies on the boundary of $\mathcal{Y}$
- Then, there exists $p \neq 0$ such that

i. For all $y \in \mathcal{Y}$, $p \cdot y \leq p \cdot y^0$,

- Proof: For the general case, see Appendix C.

Proof of Lemma 1.1-2

- If g is differentiable and $\mathcal{Y} = \{y | g(y) \geq g(y^0)\}$ is convex, then

$$y \in \mathcal{Y} \quad \Rightarrow \quad \frac{\partial g}{\partial y}(y^0) \cdot (y - y^0) \geq 0$$

$$g(y) \geq g(y^0)$$

- For $y \in \mathcal{Y} \quad \Rightarrow \quad y^\lambda = (1-\lambda)y^0 + \lambda y \in \mathcal{Y}$
- So, $g(y^\lambda) - g(y^0) \geq 0$
- Define $h(\lambda) \equiv g(y^\lambda) = g(y^0 + \lambda(y - y^0))$

Proof of Lemma 1.1-2

- For $y \in \mathcal{Y} \;\; \Rightarrow \;\; y^{\lambda} = (1-\lambda)y^{0} + \lambda y \in \mathcal{Y}$
- Define $h(\lambda) \equiv g(y^{\lambda}) = g(y^{0} + \lambda(y - y^{0}))$

$$\frac{h(\lambda) - h(0)}{\lambda} = \frac{g((y^{0} + \lambda(y - y^{0})) - g(y^{0})}{\lambda} \underline{\underline{\geq 0}}$$

- Therefore, by chain rule:

$$\begin{aligned} \frac{dh}{d\lambda}(\lambda)\bigg|_{\lambda=0} &= \frac{\partial g}{\partial y}(y^{0} + \lambda(y - y^{0})) \cdot (y - y^{0}) \\ &= \frac{\partial g}{\partial y}(y^{0}) \cdot (y - y^{0}) \underline{\underline{\geq 0}}. \end{aligned}$$

$\square$

Example

- A professor has $z = 25$ units of "brain-power"
- Allocates z_2 units to produce TSSCI papers
- Produce $y_2 = 2\sqrt{z_2}$ number of TSSCI papers
- Allocates z_3 units to produce SSCI papers
- Produce $y_3 = \sqrt{z_3}$ number of SSCI papers
- Set of feasible plans is ($y_1 = -z$)

$$\mathcal{Y} = \left\{ y \middle| g(y) = -y_1 - \frac{1}{4} y_2^2 - y_3^2 \geq 0 \right\}$$

Example

- Professor W is working at full capacity
- Professor W's output (on bdry) $y^0 = (-25, 8, 3)$
- What kind of reward scheme can support this?

$$p = -\frac{\partial g}{\partial y}(y^0) = (1, \frac{1}{2}y_2^0, 2y_3^0) = (1, 4, 6)$$

- How can you induce $(y_2^1, y_3^1) = (2, 2\sqrt{6}) \approx (2, 5)$

$$p = (1, \frac{1}{2}y_2^1, 2y_3^1) = (1, 1, 4\sqrt{6}) \approx (1, 1, 10)$$

Positive Prices (Free Disposal)

- Supporting Hyperplane theorem has economic meaning if prices are positive
  - Need another assumption
- **Free Disposal**
- For any feasible production plan $y \in \mathcal{Y}$ and any
- $\delta > 0$, the production plan $y - \delta$ is also feasible

Supporting Prices

- With free disposal, we can prove:

Proposition 1.1-3 (Supporting Prices)

- If y^0 is a boundary point of a convex set $\mathcal{Y}$
- And the free disposal assumption holds,
- Then, there exists a price vector $p > 0$ such
- that $p \cdot y \leq p \cdot y^0$ for all $y \in \mathcal{Y}$
- Moreover, if $0 \in \mathcal{Y}$, then $p \cdot y^0 \geq 0$
- Finally, for all $y \in \text{int}\mathcal{Y}$, $p \cdot y < p \cdot y^0$ - part (ii)

Supporting Prices

- **Proof:** Supporting Hyperplane Theorem says:
- There is some $p \neq 0$ such that, for all $y \in \mathcal{Y}$,
- $p \cdot (y^0 - y) \geq 0$. Now need to show $p_i \geq 0$
- By free disposal, $y' = y^0 - \delta \in \mathcal{Y}$ for all $\delta > 0$
- Setting $\delta = (1, 0, \cdots, 0)$, $p \cdot (y^0 - y') = p_1 \geq 0$
- Setting $\delta = (0, 1, 0, \cdots)$, $p \cdot (y^0 - y') = p_2 \geq 0$
- ...
- Setting $\delta = (0, \cdots, 0, 1)$, $p \cdot (y^0 - y') = p_n \geq 0$

Supporting Prices

- Since $p \cdot y \le p \cdot y^0$ for all $y \in \mathcal{Y}$, if $0 \in \mathcal{Y}$
- Set $y = 0$ and we have $p \cdot y^0 \ge 0$

- Finally, for all $y \in \text{int}\mathcal{Y}$, $p \cdot y < p \cdot y^0$ - part (ii)
- For $y \in \text{int}\mathcal{Y} \;\Rightarrow\; \exists\; y' = y + \epsilon \in \mathcal{Y}, \epsilon \gg 0$
- And $p \cdot y' = p \cdot y + p \cdot \epsilon \le p \cdot y^0$
- Since $p > 0$, we have

$$p \cdot \epsilon > 0 \Rightarrow \; p \cdot y < p \cdot y^0$$

Back to Publication Rewards

- Should NTU really pay NT$300,000 per article published in Science or Nature?
  - Is the production set for Science/Nature convex?
- What would be a better incentive scheme to encourage publications in Science/Nature?
  - Efficient Wages (High Fixed Wages)?
  - Tenure?
  - Endowed Chair Professorships?

Back to Publication Rewards

- What are some tasks do you expect piece-rate incentives to work?
  - Sales
  - Real estate agents
- What about a fixed payment?
  - Secretaries and Office Staff
  - Store Clerk
- What about other incentives schemes?
  - That's for you to answer (in contract theory)!

Summary of 1.1

- Input = Negative Output
- Vector space of y
- Convexity (quasi-concavity) is the key for supporting prices (=linearization)
- What is a good incentive scheme to induce professor to publish in Science and Nature?
- Consumer = Producer
- Homework: Exercise 1.1-4 (Optional: 1.1-6)