Games of Incomplete Information (資訊不全賽局)

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(Lecture 9, Micro Theory I)

Simultaneous Move Games – An Example

- One or more players know preferences only probabilistically (cf. Harsanyi, 1976-77)
- Cournot Duopoly Game (with private costs)
- Firm $i \in \mathfrak{I}$ of Type $\theta_i \in \Theta_i = \{\theta_{i1}, \cdots, \theta_{im_i}\}$
- Two firms; firm i of type θ_i has unit cost: $c(\theta_i)$
  - Private information: Know own cost only...
- Choose output q_i; market clearly price is:

$$p(q) = a - q_1 - q_2$$

Simultaneous Move Games – An Example

- For output vector q, firm i's profit is

$$u_i(q_i, q_{-i}, \theta_i) = (a - q_i - q_{-i} - c(\theta_i)) \cdot q_i$$

- Not knowing other's type, firms maximizes

$$U_i(q_i, \theta_i) = \underset{q_{-i}}{\mathrm{E}} \{u_i(q_i, q_{-i}, \theta_i)\}$$

$$= (a - q_i - E\{q_{-i}\} - c(\theta_i)) \cdot q_i$$

- Optimal quantity is

$$q_i(\theta_i) = \arg\max_{q_i} \{U_i(q_i, \theta_i)\}$$

Simultaneous Move Games – An Example

- FOC is both necessary and sufficient since U_i is strictly concave

$$\frac{\partial U_i}{\partial q_i} = a - E\{q_{-i}\} - c(\theta_i) - 2q_i = 0, \quad i = 1, 2.$$

- Therefore,

$$q_i^{BR}(\theta_i) = \frac{1}{2}(a - E\{q_{-i}\} - c(\theta_i)), \quad i = 1, 2.$$

- Note this depends on beliefs about others!

Simultaneous Move Games – An Example

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Simultaneous Move Games – An Example

$$q_i^{BR}(\theta_i) = \frac{1}{2}(a - E\{q_{-i}\} - c(\theta_i)), \quad i = 1, 2.$$

- Firm 2's cost is known; firm 1's cost is private
- First-Order Belief: Suppose firm 2 believes firm 1's cost is higher than previous estimates
  - Firm 2: $E\{q_1\}$ decreases; q_2 increases
- Second-Order Belief: Suppose firm 1 believes firm 2 thinks 1's cost is higher than estimates
  - Firm 1: $E\{q_2\}$ increases; q_1 decreases

Simultaneous Move Games – An Example

$$q_i^{BR}(\theta_i) = \frac{1}{2}(a - E\{q_{-i}\} - c(\theta_i)), \quad i = 1, 2.$$

- Third-Order Belief: Suppose firm 2 believes firm 1 believes that firm 2 thinks 1's cost is higher than previous estimates
  - Firm 2: $E\{q_1\}$ decreases; q_2 increases
- When does this cycle "end"?
- If there is Common Knowledge of Beliefs about

$$\text{Joint pdf } f(\theta_1, \cdots, \theta_I) \text{ over } \Theta = \times_{i \in \mathfrak{I}} \Theta_i$$

Simultaneous Move Games – An Example

- Now assume types are iid (common knowledge)

$$f(\theta_1, \theta_2) = g(\theta_1)g(\theta_2)$$

- Want to find equilibrium pure strategy:

$$q_i(\theta_i) : \Theta_i \rightarrow A_i = \mathbb{R}_+$$

- Compute expectation of BR:

$$q_i^{BR}(\theta_i) = \frac{1}{2}\left(a - \mathop{\mathrm{E}}_{\theta_{-i}}\{q_{-i}(\theta_{-i})\} - c(\theta_i)\right)$$

$$\mathop{\mathrm{E}}_{\theta_i}\left\{q_i^{BR}(\theta_i)\right\} = \frac{1}{2}\left(a - \mathop{\mathrm{E}}_{\theta_{-i}}\{q_{-i}(\theta_{-i})\} - \mathop{\mathrm{E}}_{\theta_i}\left\{c(\theta_i)\right\}\right)$$

Simultaneous Move Games – An Example

- Identical cost functions → Identical $\overline{c} = \underset{\theta_i}{\mathrm{E}}\left\{c(\theta_i)\right\}$
- Symmetric type distribution → Symmetric Eq.

$$\underset{\theta_i}{\mathrm{E}}\{q_i(\theta_i)\} = \frac{1}{2}\left(a - \underset{\theta_{-i}}{\mathrm{E}}\{q_{-i}(\theta_{-i})\} - \overline{c}\right) = \frac{1}{3}(a - \overline{c})$$

- So:

$$q_i^{BR}(\theta_i) = \frac{1}{2}\left(a - \underset{\theta_{-i}}{\mathrm{E}}\{q_{-i}(\theta_{-i})\} - c(\theta_i)\right)$$

$$= \frac{1}{2}\left(a - \frac{1}{3}(a - \overline{c}) - c(\theta_i)\right) = \frac{1}{3}a + \frac{1}{6}\overline{c} - \frac{1}{2}c(\theta_i)$$

Simultaneous Move Games – An Example

- Since $q_i^{BR}(\theta_i) = \frac{1}{3}a + \frac{1}{6}\overline{c} - \frac{1}{2}c(\theta_i)$
- Intuitively, $$q_i^{BR}(\theta_i) = \frac{1}{3}(a - \overline{c}) - \frac{1}{2}\left(c(\theta_i) - \overline{c}\right)$$ $$= \mathop{\mathrm{E}}_{\theta_i}\left\{q_i^{BR}\right\} - \frac{1}{2}\left(c(\theta_i) - \overline{c}\right)$$
- Demand is: $$p(\theta) = a - q_1(\theta) - q_2(\theta)$$ $$= \frac{1}{3}(a - \overline{c}) + \frac{1}{2}\left[c(\theta_1) + c(\theta_2)\right]$$
- Expected Price: $\overline{p} = \frac{1}{3}a + \frac{2}{3}\overline{c}$

Simultaneous Move Games – General Case

- Player $i \in \mathfrak{I}$ has Type $\theta_i \in \Theta_i = \{\theta_{i1}, \cdots, \theta_{im_i}\}$
- Set of feasible Actions: A_i
- Set of all probability measures on $\Delta(A_i)$: S_i

$$\mathcal{S} = S_1 \times \cdots \times S_I, \ \ I = |\mathfrak{I}|$$

- Strategy s_i for Player $i \in \mathfrak{I}$ of type $\theta_i \in \Theta_i$ is the function $s_i = s(\theta_i), \ \ s : \Theta_i \to S_i$
- Strategy Profile (of all players): $s = (s_1, \cdots, s_I)$

Bayesian Nash Equilibrium (BNE)

- Let $u_i(s;\theta_i), s \in \mathcal{S}$ be the payoffs of player $i \in \mathfrak{I}$
- If his type is $\theta_i \in \Theta_i$ and strategy profile is $s \in \mathcal{S}$
- Let $f(\theta_1, \cdots, \theta_I)$ be the joint distribution over types, which is common knowledge. Then, a
- strategy profile is a Bayesian Nash equilibrium
- If for each $\theta_i \in \Theta_i, i \in \mathfrak{I}, s(\theta_i)$ is a BR
  - given the common knowledge beliefs

Bayesian Nash Equilibrium (BNE)

- As if Nature moves in stage 0 to choose
  - Player types $(\theta_1, \cdots, \theta_I) \in \Theta_1 \times \cdots \times \Theta_I$
- Nature's payoffs are the same for all outcomes
- It is a BR to play mixed strategy $f(\theta_1, \cdots, \theta_I)$
- BNE of the I-player game is the NE of the $(I+1)$-player game (with Nature moving first)
  - All existence theorems apply…

Sealed First-Price and Second Price Auctions

- Bidding game with one single item for sale
- n risk neutral buyers
- Value is continuously distributed on the unit interval with cdf $F(\theta_i) \sim [0, 1]$
  - All this is common knowledge
- In Auction games,
- Buyer's type = Value (private information)
- Pure Strategy = Bid function $b = b_i(\theta_i)$

Sealed Second-Price Auction

- Each buyer submits one sealed bid

$$b_i \geq 0, i = 1, \cdots, n$$

- Buyer who makes highest bid is the winner
  - If there is a tie, the winner is chosen randomly from the tying high bidders
- The winning bidder pays the second-highest bid and receivers the item

➢ Bidding one's value is a dominant strategy!

Dominance in Sealed Second-Price Auction

- Bidding one's value is a dominant strategy
  - Note: This is independent of iid, $\#$ of bidders,...
- Proof: Consider maximum of all other bids m
- If buyer i deviates to $x < \theta_i$

1. For $m < x$: Buyer i still wins and still pays m
2. For $m > \theta_i$: Buyer i still loses (both the same)
3. For $x < m < \theta_i$: Buyer i now loses (could win)

- Similar if buyer i deviates to $x > \theta_i$ (homework)

BNE in Sealed Second-Price Auction

- Bidding one's value is a dominant strategy
- Consider the order statistics of values (highest to lowest): $\{\theta_{(1)}, \theta_{(2)}, \cdots, \theta_{(n)}\}$
- In this (dominance-solvable) BNE:
- Winner is buyer with value $\theta_{(1)}$ and pays $\theta_{(2)}$
- Buyer i's expected payment conditional on winning is $E\{\theta_{(2)} | \theta_{(1)} = \theta_i\}$
  - Note: There are other "crazy" asymmetric BNE...

Sealed First Price Auction

- Each buyer submits one sealed bid

$$b_i \geq 0, i = 1, \cdots, n$$

- Buyer who makes highest bid is the winner
  - If there is a tie, the winner is chosen randomly from the tying high bidders
- The winning bidder pays his bid and receivers the item

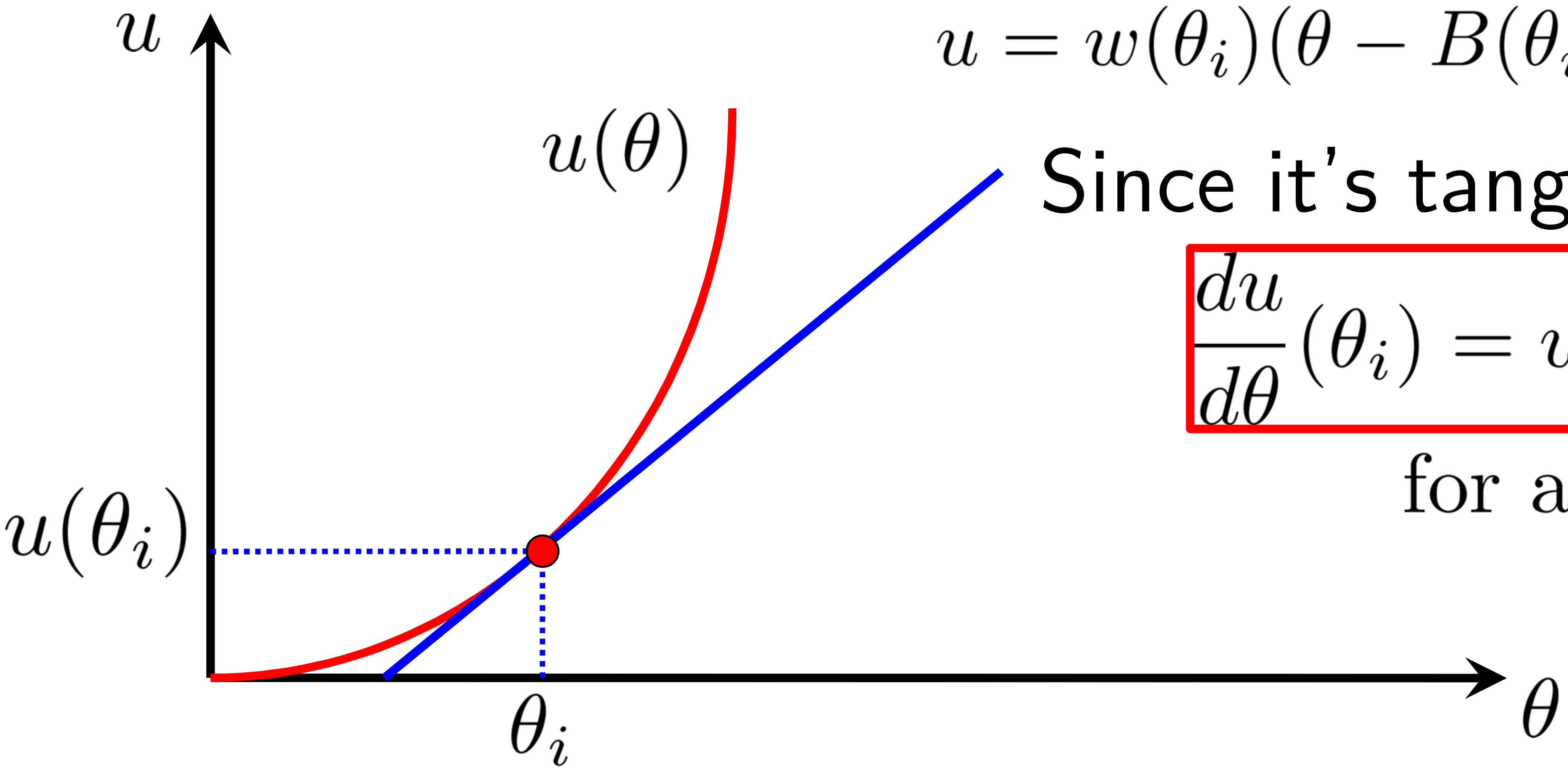
Sealed First Price Auction

- Assume buyer values are iid $F(\theta) \sim [0, \beta]$
- Solve for Equilibrium Bidding Strategy $B(\theta_i)$
  - Symmetric (since we assume iid values)
  - Strictly increasing (high types unlikely to bid low)
  - Assume $B(\theta_i) \in C^1$ (by assuming $F(\theta) \in C^1$)
- If others follow BNE, the win probability of following BNE is $w(\theta_i) = F^{n-1}(\theta_i)$
  - Win only when you are the highest type...

Sealed First Price Auction

- For Equilibrium Payoff $u(\theta_i) = w(\theta)(\theta - B(\theta))$
- If deviate to bidding a fixed $B(\theta_i)$, payoff is:

$$u = w(\theta_i)(\theta - B(\theta_i)) \leq u(\theta)$$



Since it's tangent at θ_i

$$\boxed{\frac{du}{d\theta}(\theta_i) = w(\theta_i)}$$

for all $\theta_i \in [0, \beta]$

Sealed First Price Auction

- Since $B(0)=0$, we have $w(0)=0$ and $u(0)=0$
- Thus, $\frac{du}{d\theta}(\theta_i)=w(\theta_i)$ becomes:

$$u(\theta_i)=\int_0^{\theta_i}\frac{du}{d\theta}d\theta=\int_0^{\theta_i}w(\theta)d\theta$$

$$=\theta_i w(\theta_i)-\int_0^{\theta_i}\theta dw(\theta) \quad =w(\theta_i)(\theta_i-B(\theta_i))$$

Same as Second-Price!!

- Thus, $B(\theta_i)=\frac{\int_0^{\theta_i}\theta dw(\theta)}{w(\theta_i)}=\mathrm{E}\left\{\theta_{(2)}\middle|\theta_{(1)}=\theta_i\right\}$

Prop. 10.1-1/2: Revenue/Buyer Equivalences

- In an n-bidder auction where bidders are risk neutral and values are iid $F(\theta) \sim [0, \beta], F \in C^1$
- For the sealed first- and second-price auctions,
- Proposition 10.1-1: The equilibrium expected revenue is the same.
  - In fact, we have Buyer Equivalence as well!
- Proposition 10.1-2: The equilibrium payoff for each buyer type is the same.

Prop. 10.1-3/4: Strategic Equivalences

- Dutch Auction
- English Auction
- Prop. 10.1-3: Equilibrium bidding strategies of the FP and Dutch auctions are the same
- Prop. 10.1-4: Equilibrium bidding strategies of the SP and English auctions are the same

- Note: Not just revenue, and assumption free!

Sequential Move Games

- Player i_t moves in stage t has Type $\theta_t \in \Theta_t$
- Set of feasible Actions: A_t
- Set of all probability measures on $\Delta(A_t)$: S_t

$$\mathcal{S} = S_1 \times \cdots \times S_T$$

- Strategy s_t for Player i_t of type $\theta_t \in \Theta_t$ is the function $s_t = s(h^t, \theta_t), \quad s : H^t \times \Theta_t \to S_t$
- Strategy Profile (of all players): $s = (s_1, \cdots, s_T)$

BNE in Sequential Move Games

- Let $u_t(s;\theta_t), s \in \mathcal{S}$ be the payoffs of player i_t
- If his type is $\theta_t \in \Theta_t$ and strategy profile is $s \in \mathcal{S}$
- Let $f(\theta_1, \cdots, \theta_T)$ be the joint distribution over types, which is common knowledge. Then, a
- strategy profile is a Bayesian Nash equilibrium
- If for each t and $\theta_t \in \Theta_t, s_t(h^t, \theta_t)$ is a BR
  - given the common knowledge beliefs
- Note: Assume independent types for 10.1…

Sequential Move Games – An Example

- Cournot Duopoly Game (with private costs)
- Two firms; firm i has unit cost: $c_i, i = 1, 2$
  - Private information: Know own cost only...
- Choose output q_i; market clearly price is:

$$p(q) = a - q_1 - q_2$$

- Firm 1 moves first
- Firm 2 observes and chooses $q_i^{BR}(q_1, c_2)$
  - Anticipated by Firm 1...

Sequential Move Games – An Example

- Firm 1 forms belief $E_1\{c_2\}$
- For output vector q, firm 2's profit is

$$U_2(q, c_2) = (a - q_1 - q_2 - c_2) \cdot q_2$$

- Firm 2's optimal quantity is

$$q_2^{BR}(q_1, c_2) = \arg\max_{q_2} \{U_2(q, c_2)\} = \frac{1}{2}(a - c_2 - q_1)$$

- Firm 1's belief is $q_2^{BR}(\overline{c}_{12}) = \dfrac{1}{2}(a - E_1\{c_2\} - q_1)$

Sequential Move Games – An Example

- Firm 1's belief is $q_2^{BR}(\overline{c}_{12}) = \frac{1}{2}(a - E_1\{c_2\} - q_1)$
- Profit: $U_1 = (a - q_1 - q_2^{BR}(\overline{c}_{12}) - c_1)q_1$

$$= \left[\frac{1}{2}(a - q_1) + \frac{1}{2}E_1(c_2) - c_1\right] q_1$$

- Maximized at: $q_1^{BR} = \frac{1}{2}a - c_1 + \frac{1}{2}E_1\{c_2\}$
- So, $q_2^{BR}(q_1, c_2) = \frac{1}{2}(a - c_2 - q_1)$

$$= \frac{1}{4}a + \frac{1}{2}(c_2 - c_1) - \frac{1}{4}E_1(c_2)$$

- (3 firms?)

Summary of 10.1

- Bayesian Games
  - Incomplete Information as "Types"
- Bayesian Nash Equilibrium
- Auction Games:
  - First-Price (Dutch) vs. Second-Price (English)
  - Revenue/Buyer/Strategic Equivalences

- HW 10.1: Riley – 10.1-1, 3, 4
- Do the case of $x > \theta_i$ in second-price auctions