Games with History

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(Lecture 7, Micro Theory I)

Finitely Repeated Game

- Play the same simultaneous game each stage
- History of the game: h_i^t

$=$ all information available to player i at period t

- The Second Stage Strategy is a function of history h_i^2
- Two/Three stage repeated game strategy:

$$s_i = (s_i^1, s_i^2(h_i^2)) \in S_i \times S_i$$
$$s_i = (s_i^1, s_i^2(h_i^2), s_i^3(h_i^3)) \in S_i \times S_i \times S_i$$

Finitely Repeated Game

- T-stage repeated game strategy:

$$s = (s^1, \cdots, s^T) \in \mathcal{S} = S^1 \times \cdots \times S^T$$

- Stage game strategy space:

$$S^t = \times_{i \in I} S_i, t = 1, \cdots, T$$

- Player i's stage t payoff $u_i(s^t)$
- Discount rate δ
- Player i's (total) payoff $U_i(s) = \sum_{t=1}^{T} \delta^{t-1} u_i(s^t)$

Prop. 9.2-1: NE of a Finitely Repeated Game

i. Suppose each $\overline{s}^t = \{\overline{s}_1^t, \overline{s}_2^t, \cdots, \overline{s}_n^t\}, t = 1, \cdots, T$ is an NE of the t-th stage game,
Then, $\overline{s} = (\overline{s}^1, \overline{s}^2, \cdots, \overline{s}^T)$ (indep. of history) is an NE of the T-stage repeated game

ii. If $\hat{s}$ is the unique NE of the stage game, then $(\hat{s}, \cdots, \hat{s})$ is an NE of the finitely repeated game

- Proof:

Proof of Proposition 9.2-1: Part (i)

- Compare NE $\overline{s} \in \mathcal{S}$ and player i's deviation s_i
- Need to show that:

$$U_i(s_i, \overline{s}_{-i}) - U_i(\overline{s}_i, \overline{s}_{-i}) = \sum_{t=1}^{T} \delta^{t-1} \left[u_i\left(s_i^t, \overline{s}_{-i}^t\right) - u_i\left(\overline{s}_i^t, \overline{s}_{-i}^t\right) \right] \leq 0$$

- Since $\overline{s}^t$ is an NE of the t-th stage game,

$$u_i\left(s_i^t, \overline{s}_{-i}^t\right) - u_i\left(\overline{s}_i^t, \overline{s}_{-i}^t\right) \leq 0 \quad \text{for all } t$$

- Sum them up and you are done!

Proof of Proposition 9.2-1: Part (ii)

- Compare NE $\overline{s} \in \mathcal{S}$ and player i's deviation s_i in which for some t, $s_i^t \neq \overline{s}_i^t$, and $s_i^\tau = \overline{s}_i^\tau, \forall \tau \neq t$

$$U_i(s_i, \overline{s}_{-i}) - U_i(\overline{s}_i, \overline{s}_{-i}) = \delta^{t-1}\left[u_i\left(s_i^t, \overline{s}_{-i}^t\right) - u_i\left(\overline{s}_i^t, \overline{s}_{-i}^t\right)\right] \leq 0$$

- Since $\hat{s} \in \mathcal{S}$ is the unique NE of the stage game
- There is only one $\overline{s}_i^t = \hat{s}$ such that

$$u_i\left(s_i^t, \overline{s}_{-i}^t\right) - u_i\left(\overline{s}_i^t, \overline{s}_{-i}^t\right) \leq 0$$

- So, this NE must be $(\hat{s}, \cdots, \hat{s})$.

Nash Equilibrium: Repeated Partnership Game

- Consider the Partnership Game in Section 9.1...
- Two Agents have equal share in a partnership
- Choose Effort: $a_i \in A_i = \{1, 2, 3\}$
- Total revenue: $R = 12a_1a_2$
- Cost to agent i: $C_i(a_i) = a_i^3$
- Payoff: $u_i(s) = R/2 - C_i(a_i) = 6a_1a_2 - a_i^3$

- Game matrix and Nash Equilibrium...

Nash Equilibrium: Repeated Partnership Game

Two static-NE: *(1,1)*, *(2,2)*

Combo of static-NE is NE in 2-stage repeated game

Best Payoff = 16+16δ

Player 2: Colin

Player 1: Rowena	*1*	*2*	*3*
1	5, 5	11, 4	17, -9
2	4, 11	16, 16	28, 9
3	-9, 17	9, 28	27, 27

Can we do better?

Equilibrium Threats

- These are NOT the only two equilibria
- Agents can threat to play the bad equilibrium in stage 2 to induce (3,3) and earn (27, 27)…
- EX: Use: $\overline{s}_i^1 = 3,\quad \overline{s}_i^2(h^2) = 2 \text{ if } h^2 = (3,3)$

$$\overline{s}_i^2(h^2) = 1 \text{ if } h^2 \neq (3,3)$$

- If other agent follows this strategy,
- Is it a BR to follow this strategy?
- Yes for Stage 2 (both *(2, 2)* & *(1, 1)* are static-NE)
  – For Stage 1…

Nash Equilibrium: Repeated Partnership Game

$$u(\text{follow}) = 27 + \delta \cdot 16$$

$$u(\text{defect}) = 28 + \delta \cdot 5$$

Yes if $\delta \geq \frac{1}{11}$

Player 2: Colin

Player 1: Rowena

	1	*2*	*3*
1	5, 5	11, 4	17, -9
2	4, 11	16, 16	28, 9
3	-9, 17	9, 28	27, 27

What if MORE rounds?

Sequential Move Games

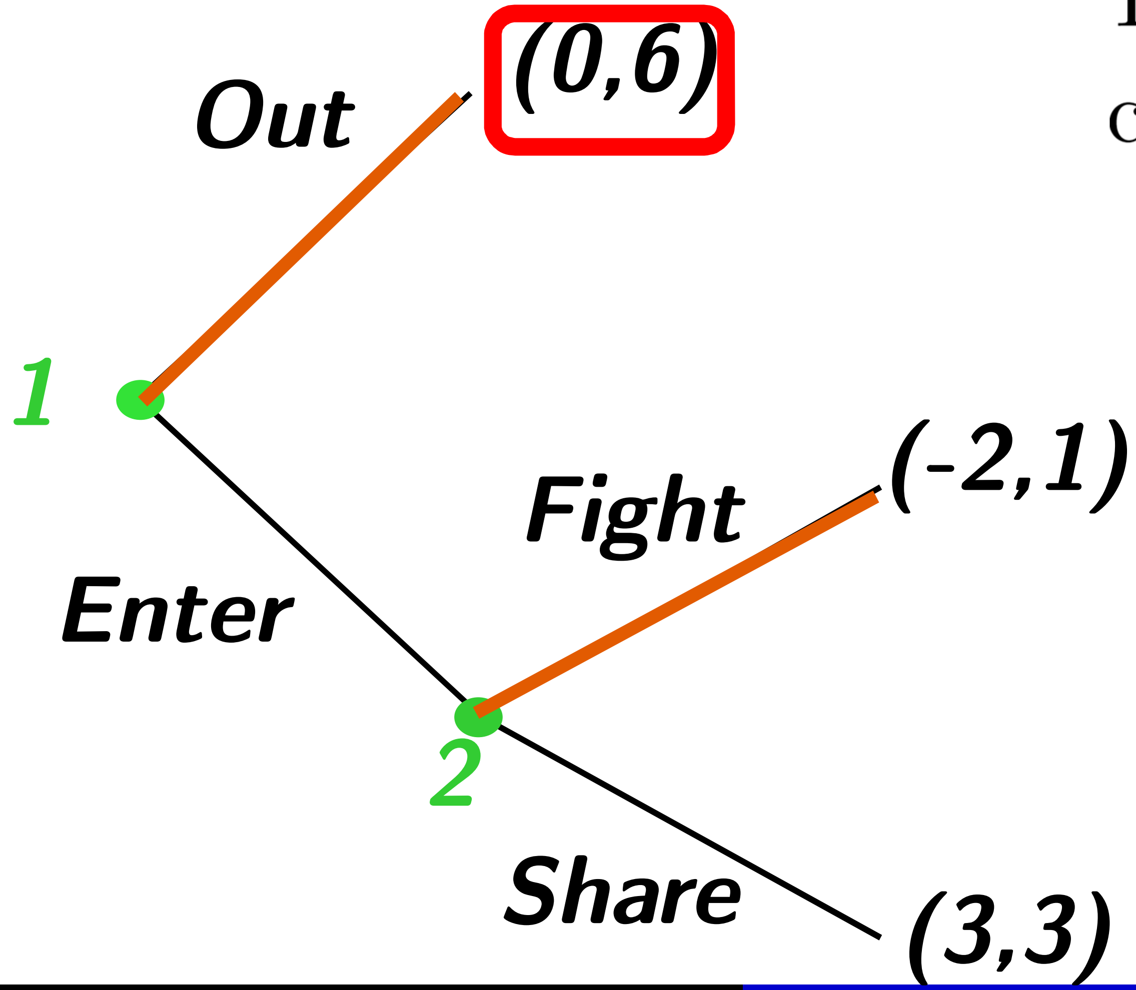
- T Stages
- Agent $i = i_t \in \mathcal{I}$ moves in stage t
- History prior to stage t observed by i : h_i^t
- Set of possible pure strategies in stage t is S_t
- Strategy Profile: $s = (s_1, \cdots, s_T)$
- (Expected) Payoffs: $u_i(s)$ depends on s

- Exists other Nash equilibrium not solved by BI...

Entry Game with Sub-game

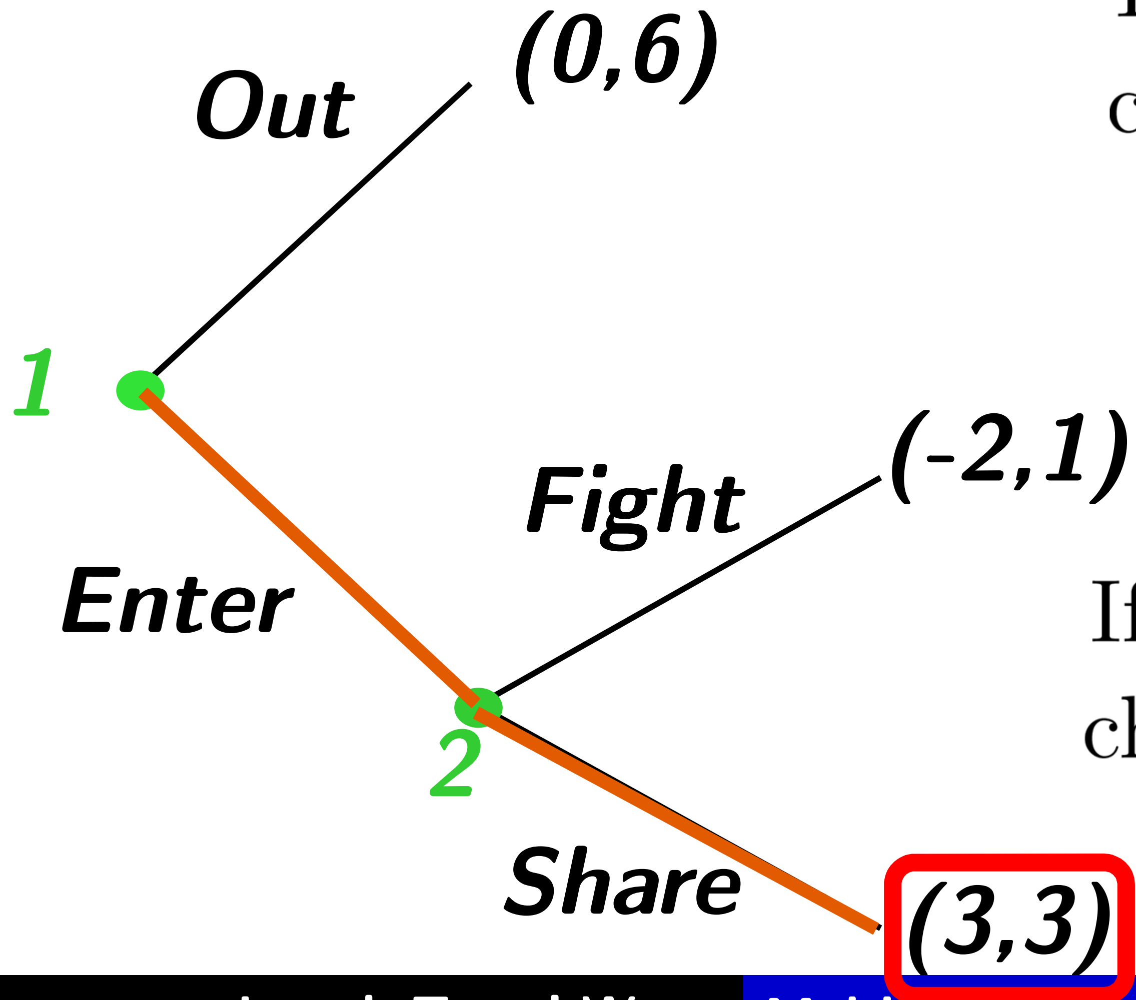
- Selten's Chain Store Paradox

If $s_2 = Fight$
choose $s_1 = Out$



Entry Game with Sub-game

- Selten's Chain Store Paradox



If $s_2 = Fight$
choose $s_1 = Out$

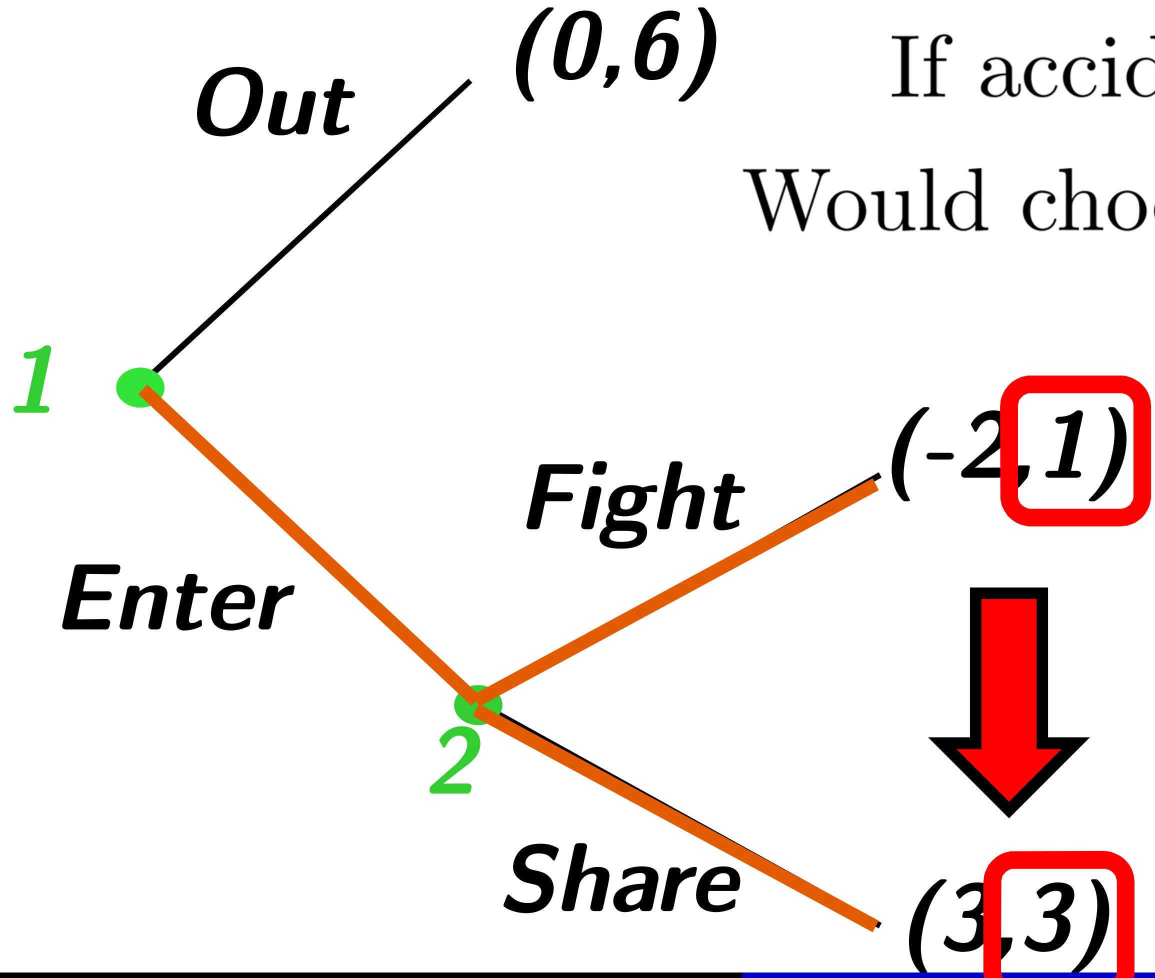
If $s_2 = Share$
choose $s_1 = Enter$

Entry Game with Sub-game

But $(Out, Fight)$ is **not credible**:

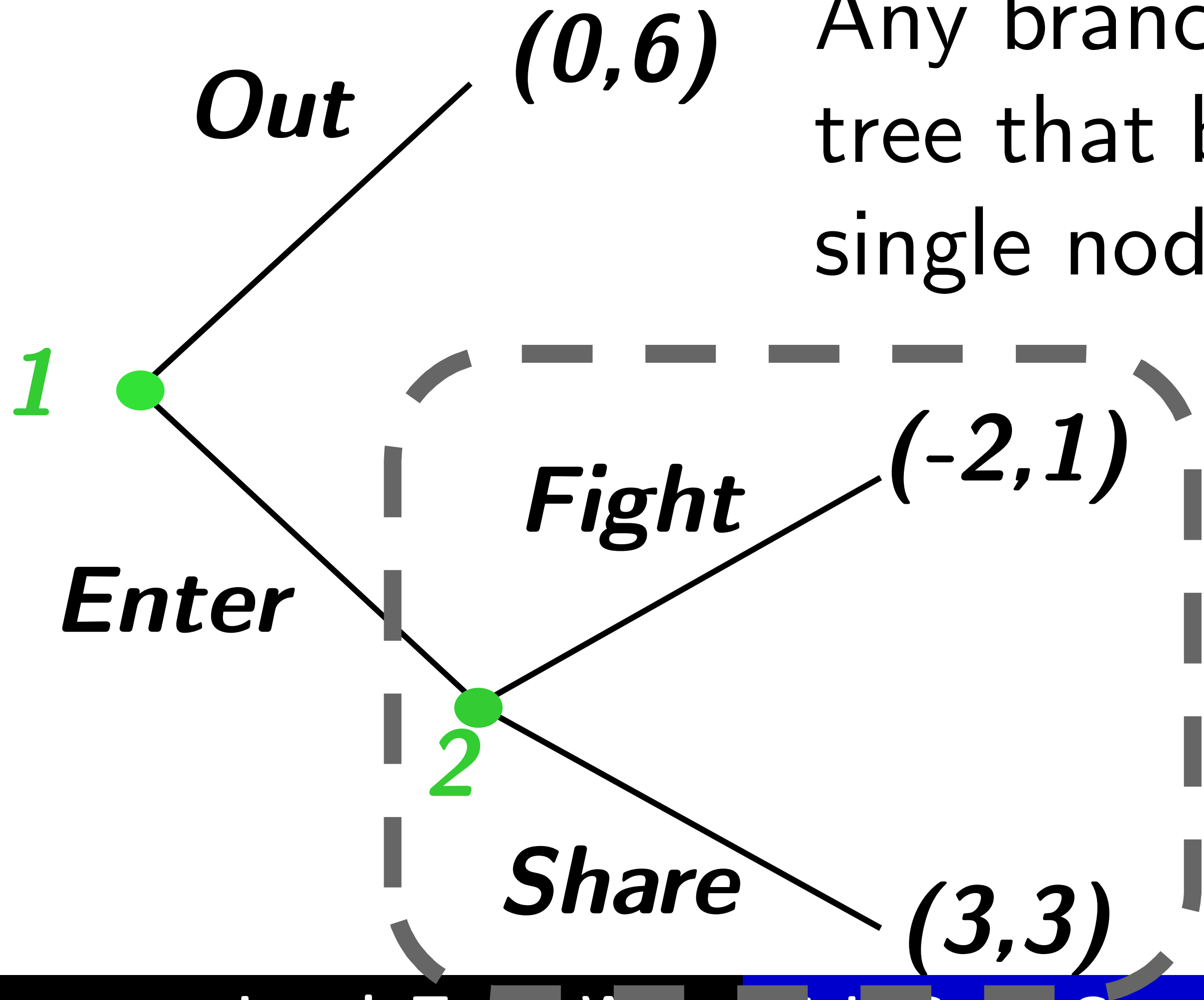
If accidentally $Enter$,

Would choose $s_2 = Share$!!



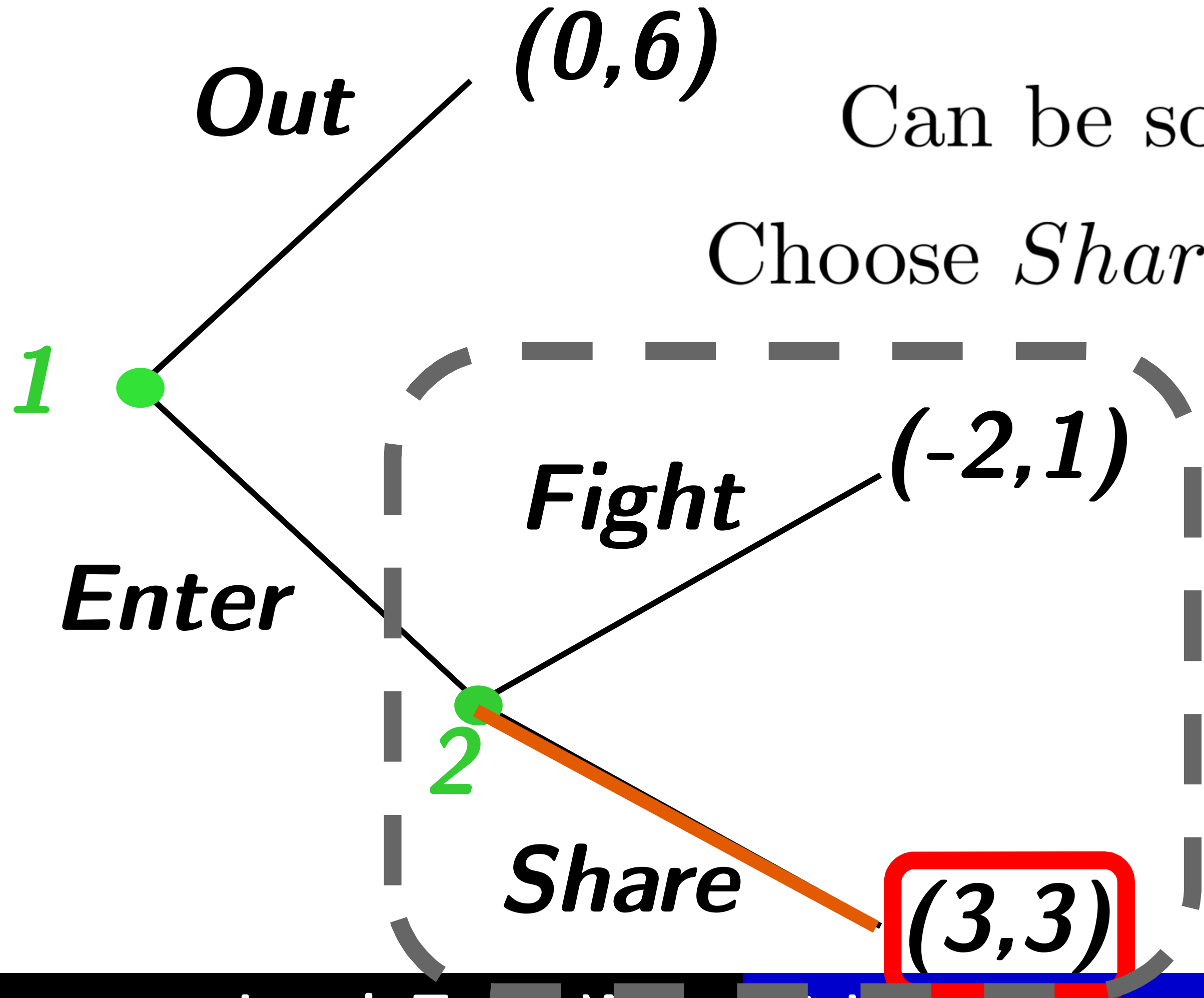
Definition of a Sub-game

Sub-game :=
Any branch of a game tree that begins with a single node



Definition of Sub-game Perfect Equilibrium

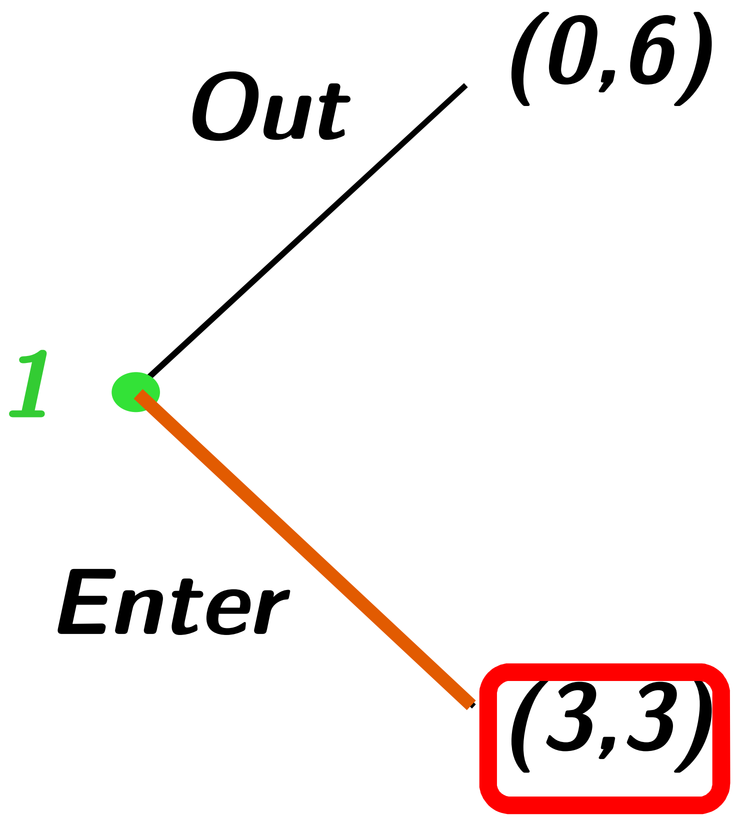
NE strategy profile is SPE if it is also NE in all sub-game



Can be solved by **BI**...

Choose *Share* is sub-game

SPE of the (Reduced) Entry Game



- Reduced entry game (with payoffs from the sub-game)

$$\text{choose } s_1 = Enter$$

- Unique SPE is *(Enter, Share)*

Prop. 9.2-2: One-Stage Deviation Principle

- In a T-stage sequential move game
- If strategy profile $\overline{s} = (\overline{s}^1, \overline{s}^2(h^2), \cdots, \overline{s}^T(h^T))$ has no profitable one-stage deviation,
- Then, this strategy profile is SPE.
- Proof:

Proof of Proposition 9.2-2

- For $\overline{s} \in \mathcal{S}$ (no profitable 1-stage deviation) and $s = (s^1, s^2(h^2), \cdots, s^T(h^T))$ (only player i deviates)
- Consider $\overline{s}(\theta)$ the hybrid of the two strategies:

$$\overline{s}(\theta)^t(h^t) = \begin{cases} \overline{s}^t(h^t) \text{ if } t \geq \theta \\ s^t(h^t) \text{ if } t < \theta \end{cases}$$

$$\overline{s}(\theta)_i^t(h^t) = \begin{cases} \overline{s}_i^t(h^t) \text{ if } t \geq \theta \\ s_i^t(h^t) \text{ if } t < \theta \end{cases}$$

- Then, since $\overline{s}$ has no profitable 1-stage deviation

$$\overline{s}(\theta) = (s^1, \cdots, s^{\theta-1}(h^{\theta-1}), \overline{s}^\theta(h^\theta), \cdots, \overline{s}^T(h^T))$$

- Has no profitable 1-stage deviation for stage $t \geq \theta$

Proof of Proposition 9.2-2

- Consider the last stage $t=a$ s. t. $\overline{s}_i^t(h^t) \neq s_i^t(h^t)$
- Then, $s = \overline{s}(a+1)$
- Claim: $U_i(\overline{s}(a)) \geq U_i(\overline{s}(a+1)) = U_i(s)$
- Since $\overline{s}$ has no profitable 1-stage deviation,

$$U_i(\overline{s}(a)) - U_i(\overline{s}(a+1))$$

$$\overline{s}(\theta)_i^t(h^t) = \begin{cases} \overline{s}_i^t(h^t) \text{ if } t \geq \theta \\ s_i^t(h^t) \text{ if } t < \theta \end{cases}$$

$$= \delta^{a-1}\left[u_i\left(\overline{s}_i^a(h^a), \overline{s}_{-i}^a(h^a)\right) - u_i\left(s_i^a(h^a), \overline{s}_{-i}^a(h^a)\right)\right]$$

$$\geq 0$$

Proof of Proposition 9.2-2

- For the next-to-last stage $t = b$ s. t. $\overline{s}_i^t(h^t) \neq s_i^t(h^t)$
- Then, $\overline{s}(b+1) = \overline{s}(a)$ (no deviations in btw)
- Claim: $U_i(\overline{s}(b)) \geq U_i(\overline{s}(b+1))$
- Since $\overline{s}$ has no profitable 1-stage deviation,

$$U_i(\overline{s}(b)) - U_i(\overline{s}(b+1))$$
$$= \delta^{b-1}\left[u_i\left(\overline{s}_i^b(h^b), \ \cdot \ \right) - u_i\left(s_i^b(h^b), \ \cdot \ \right)\right] \geq 0$$

- Thus, $U_i(\overline{s}(b)) \geq U_i(\overline{s}(b+1)) = U_i(\overline{s}(a))$
- (by induction, QED.) $\geq U_i(\overline{s}(a+1)) = U_i(s)$

Proof of Proposition 9.2-2

- You can do the same for the next-next-to-last stage $t = c$ such that $\overline{s}_i^t(h^t) \neq s_i^t(h^t)$ and so on...

$$
\begin{aligned}
U_i(\overline{s}) = U_i(\overline{s}(1)) &\geq \cdots \\
&\geq U_i(\overline{s}(c+1)) = U_i(\overline{s}(b)) \\
&\geq U_i(\overline{s}(b+1) = U_i(\overline{s}(a)) \\
&\geq U_i(\overline{s}(a+1) = U_i(s)
\end{aligned}
$$

- So $\overline{s}$ is a NE for the whole game.
- The same applies to all sub-games, so it's SPE!

Cor. 9.2-3: 1-Stage Deviation Principle for FRG

- Finitely repeated games (FRG) is a special case of sequential move games…
- In a finitely repeated game,
- If strategy profile $\overline{s} = (\overline{s}^1, \overline{s}^2(h^2), \cdots, \overline{s}^T(h^T))$

  has no profitable one-stage deviation,

  Then, this strategy profile is SPE.

- Proof: Special case of Proposition 9.2-2.

Summary of 9.2

- Finitely Repeated Games
  - Equilibrium Threat and Efficiency
- Sequential Move Game
- Sub-game Perfect Equilibrium
  - Solved by Backward Induction
- HW 9.2: Riley – 9.2-2 and 9.2-3 and BGT5