

Supporting Prices and Convexity

Joseph Tao-yi Wang

2012/9/12

(Lecture 1, Micro Theory I)

Overview of Chapter 1

- Theory of Constrained Maximization
 - Why should we care about this?
- What is Economics?
- Economics is **the study of how society manages its scarce resources** (Mankiw, Ch.1)
 - “Economics is the science which studies human behavior as a relationship between given ends and scarce means which have alternative uses.”
([Lionel Robbins](#), 1932)

Overview of Chapter 1

- Other Historical Accounts:
 - Economics is the “study of how societies use scarce resources to produce valuable commodities and distribute them among different people.”
([Paul A. Samuelson](#), 1948)
- My View: Economics is a study of institutions and human behavior (reactions to institutions)
- Either way, constrained maximization is key...

Tools Introduced in Chapter 1

1. Supporting Hyperplanes (and Convexity)
 2. First Order Conditions (Kuhn-Tucker)
 3. Envelope Theorem
- But why do I need to know the math?
 - Because math provides **precise definitions** and make **precise predictions** for those “it depends”...
 - The financial crisis tells us what happens when you ignore all the assumptions and conditions required for your theory to work...

Publication Reward Problem

- Example: How should NTU reward its professors to publish journal articles?
 - Should NTU pay, say, NT\$300,000 per article published in Science or Nature?
- Well, **it depends...**
- Peek the answer ahead:
 - Yes, if the production set is convex.
 - No, if, for example, there is initial increasing returns to scale.

Supporting Prices

- More generally, can **prices** and **profit** maximization provide appropriate incentives for all **efficient production plans**?
 - Is there a price vector that **supports** each efficient production plan?
- (Yes, but when?)
- Need some definitions first...

Production Plant

- A production plant can:
- produce n outputs $q = (q_1, \dots, q_n)$
- using up to m inputs $z = (z_1, \dots, z_m)$
- **Production Plan** (z, q)
- **Production Set** $Y \subset \mathbf{R}_+^{m+n}$
=Set of all Feasible Production Plan
- **Production Vector** (treat inputs as negative)
 $y = (-z, q) = (-z_1, \dots, -z_m, q_1, \dots, q_n)$

Production Set and Profits

- Production vector

$$y = (y_1, \dots, y_{m+n}) = (-z_1, \dots, -z_m, q_1, \dots, q_n)$$

- Production Set $\mathcal{Y} \subset \mathbf{R}^{m+n}$

=Set of Feasible Production Plan

- Price vector $p = (p_1, \dots, p_{m+n})$

$$\bullet \text{ Profit } \Pi = \underbrace{\sum_{i=m+1}^{m+n} p_i y_i}_{\text{total revenue}} - \underbrace{\sum_{i=1}^m p_i z_i}_{\text{total cost}} = p \cdot y$$

EX: Production Function and Production Set

- A professor has 25 units of “brain-power”
- Allocates z_1 units to produce TSSCI papers
- Produce $q_1 = 4\sqrt{z_1}$ (**Production Function**)

- **Production Set**

$$Y_1 = \{(z_1, q_1) \mid z_1 \geq 0, q_1 \leq 4\sqrt{z_1}\}$$

- Treating inputs as negatives, $y = (-z, q)$
- Production Set is

$$\mathcal{Y}_1 = \{(y_1, y_2) \mid -16y_1 - y_2^2 \geq 0\}$$

Production Efficiency

- A production plan y is **wasteful** if another plan in \mathcal{Y} achieves **larger** output with smaller input
- \bar{y} is **production efficient** (=non-wasteful) if

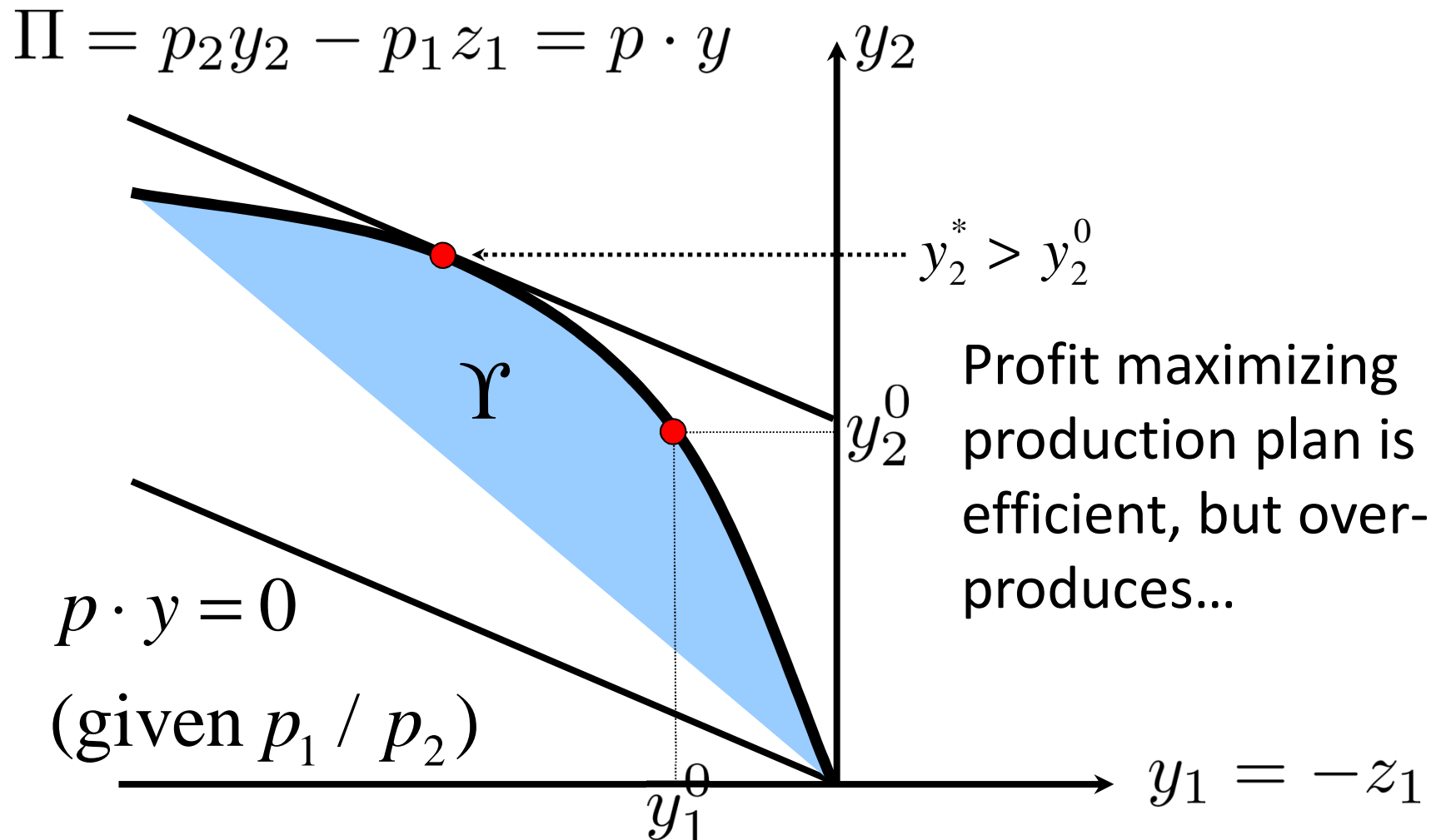
There is no $y \in \mathcal{Y}$ such that $y > \bar{y}$

- Note: $y \geq \bar{y}$ if $y_j \geq \bar{y}_j$ for all j
- $y > \bar{y}$ if inequality is strict for some j
- $y \gg \bar{y}$ if inequality is strict for all j

EX: Can Prices Support Efficient Production?

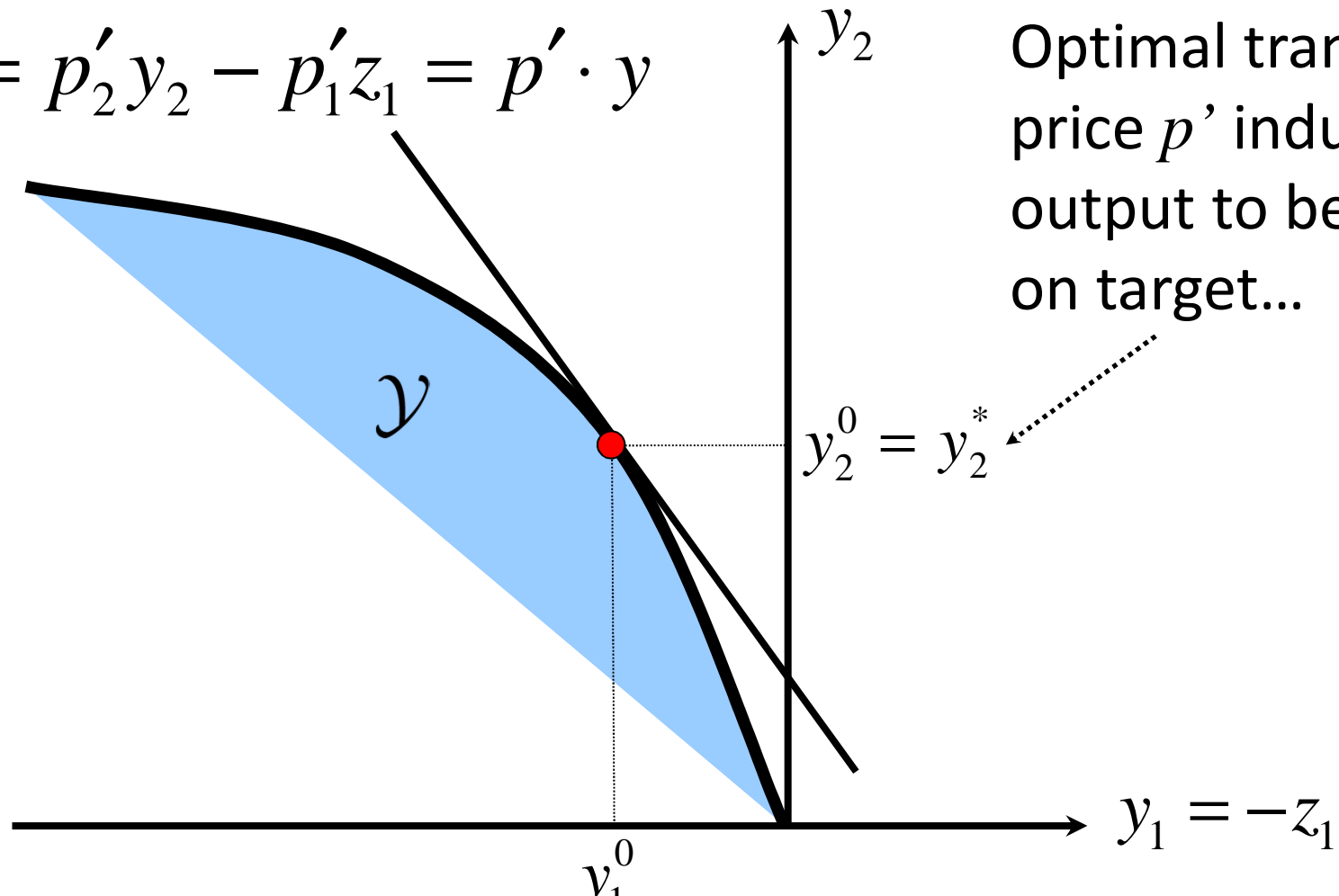
- A professor has 25 units of “brain-power”
- Allocates y_1 units to produce TSSCI papers
- Price of brain-power is p_1
- Production Set \mathcal{Y}_1
- Can we induce production target y_2^0 ?
- With piece-rate prize p_2 ?

Can Prices Support Efficient Production Plan?

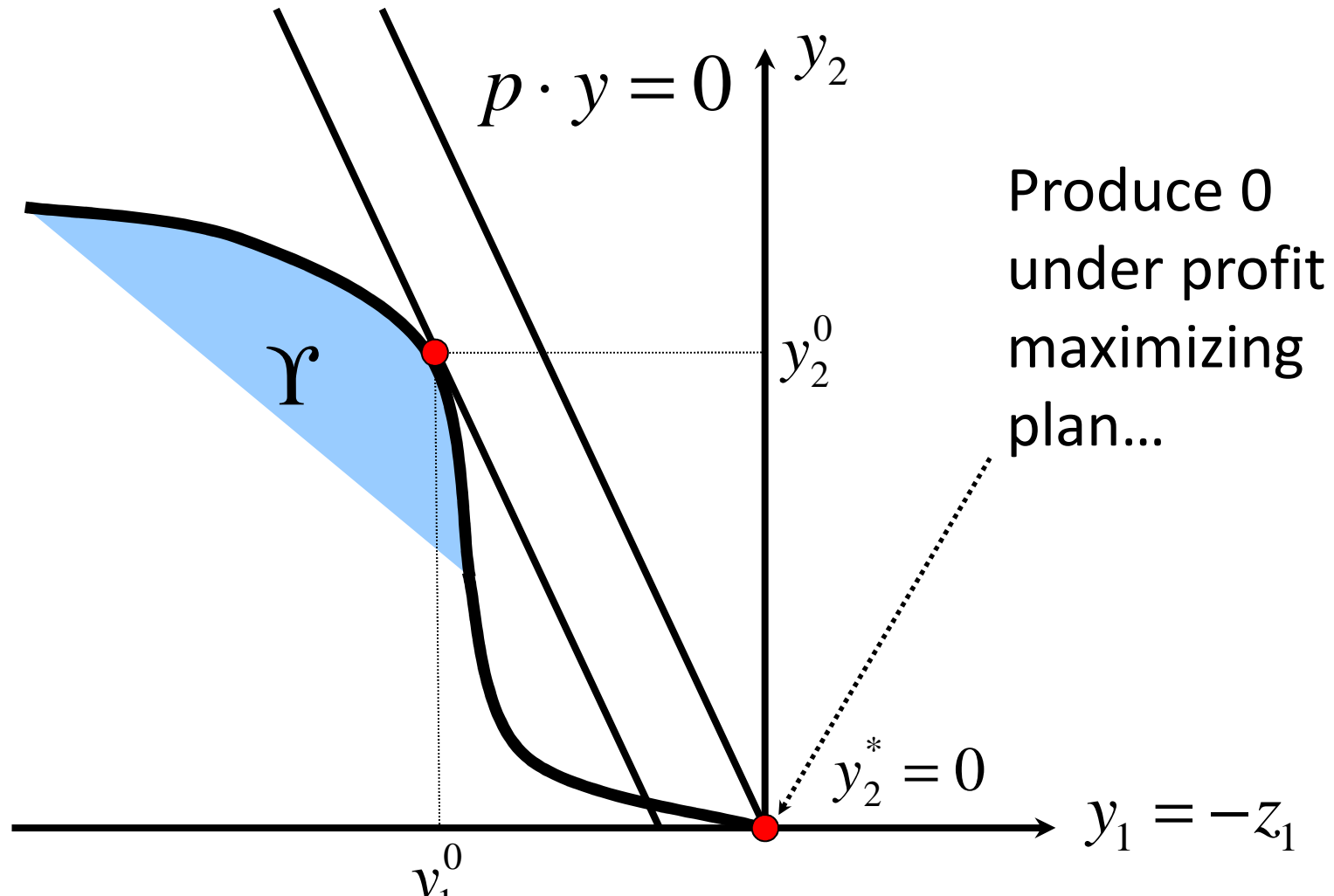


Too High? Let's Lower the Transfer Price...

$$\Pi = p'_2 y_2 - p'_1 z_1 = p' \cdot y$$



Will this Always Work?



What Made It Fail?

- The last production set was NOT **convex**.
- Production Set \mathcal{Y}_1 is **convex** if for any y^0, y^1
- Its **convex combination** (for $0 < \lambda < 1$)
$$y^\lambda = (1 - \lambda)y^0 + \lambda y^1 \in \mathcal{Y}_1$$
- (is also in the production set)
- Is it true that we can use prices to guide production decisions as long as production sets are convex?

Supporting Hyperplane Theorem

Proposition 1.1-1 (**Supporting Hyperplane**)

- Suppose $\mathcal{Y} \subset \mathbf{R}^n$ is non-empty and convex,
- And y^0 lies on the boundary of \mathcal{Y}
- Then, there exists $p \neq 0$ such that
 - i. For all $y \in \mathcal{Y}$, $p \cdot y \leq p \cdot y^0$, and
 - ii. For all $y \in \text{int}\mathcal{Y}$, $p \cdot y < p \cdot y^0$
- Proof: For the general case, see Appendix C.

Special Case of Supporting Hyperplane Thm

- Consider special case where
- Production set \mathcal{Y} is the **upper contour set**
 $\mathcal{Y} = \{y \mid g(y) \geq g(y^0)\}$, g is differentiable
- Suppose the gradient vector is non-zero at y^0
- The **linear approximation** of g at y^0 is:

$$\bar{g}(y) = g(y^0) + \frac{\partial g}{\partial y}(y^0) \cdot (y - y^0)$$

- If \mathcal{Y} is convex, it lies in \bar{g} 's upper contour set

Special Case of Supporting Hyperplane Thm

- **Lemma 1.1-2**

- If g is differentiable and $\mathcal{Y} = \{y \mid g(y) \geq g(y^0)\}$ is convex, then

$$y \in \mathcal{Y} \Rightarrow \frac{\partial g}{\partial y}(y^0) \cdot (y - y^0) \geq 0$$
$$g(y) \geq g(y^0)$$

-
- This tells us how to calculate the supporting prices (under this special case):

- For boundary point y^0 , choose $p = -\frac{\partial g}{\partial y}(y^0)$

From Lemma to Supporting Hyperplane Thm

- If g is differentiable and $\mathcal{Y} = \{y \mid g(y) \geq g(y^0)\}$ is convex, then

$$y \in \mathcal{Y} \Rightarrow -p \cdot (y - y^0) \geq 0$$

$$\Rightarrow p \cdot y \leq p \cdot y^0$$

- This gives us part (i) of S. H. T.
- For $y \in \text{int}\mathcal{Y} \Rightarrow \exists y' = y + \epsilon \in \mathcal{Y}, \epsilon \gg 0$
- And $p \cdot y' = p \cdot y + p \cdot \epsilon \leq p \cdot y^0$
- Since $p \cdot \epsilon > 0 \Rightarrow p \cdot y < p \cdot y^0$

Supporting Hyperplane Theorem

Proposition 1.1-1 (**Supporting Hyperplane**)

- Suppose $\mathcal{Y} \subset \mathbf{R}^n$ is non-empty and convex,
- And y^0 lies on the boundary of \mathcal{Y}
- Then, there exists $p \neq 0$ such that
 - i. For all $y \in \mathcal{Y}$, $p \cdot y \leq p \cdot y^0$, and
 - ii. For all $y \in \text{int}\mathcal{Y}$, $p \cdot y < p \cdot y^0$
- Proof: For the general case, see Appendix C.

Proof of Lemma 1.1-2

- If g is differentiable and $\mathcal{Y} = \{y \mid g(y) \geq g(y^0)\}$ is convex, then

$$y \in \mathcal{Y} \Rightarrow \frac{\partial g}{\partial y}(y^0) \cdot (y - y^0) \geq 0$$
$$g(y) \geq g(y^0)$$

- For $y \in \mathcal{Y} \Rightarrow y^\lambda = (1 - \lambda)y^0 + \lambda y \in \mathcal{Y}$
- So, $g(y^\lambda) - g(y^0) \geq 0$
- Define $h(\lambda) \equiv g(y^\lambda) = g(y^0 + \lambda(y - y^0))$

Proof of Lemma 1.1-2

- For $y \in \mathcal{Y} \Rightarrow y^\lambda = (1 - \lambda)y^0 + \lambda y \in \mathcal{Y}$
- Define $h(\lambda) \equiv g(y^\lambda) = g(y^0 + \lambda(y - y^0))$

$$\frac{h(\lambda) - h(0)}{\lambda} = \frac{g((y^0 + \lambda(y - y^0))) - g(y^0)}{\lambda} \geq 0$$

- Therefore, by chain rule:

$$\begin{aligned} \left. \frac{dh}{d\lambda}(\lambda) \right|_{\lambda=0} &= \frac{\partial g}{\partial y}(y^0 + \lambda(y - y^0)) \cdot (y - y^0) \\ &= \frac{\partial g}{\partial y}(y^0) \cdot (y - y^0) \geq 0. \quad \square \end{aligned}$$

Example

- A professor has $z = 25$ units of “brain-power”
- Allocates z_2 units to produce TSSCI papers
- Produce $y_2 = 2\sqrt{z_2}$ number of TSSCI papers
- Allocates z_3 units to produce SSCI papers
- Produce $y_3 = \sqrt{z_3}$ number of SSCI papers
- Set of feasible plans is $(y_1 = -z)$

$$\mathcal{Y} = \left\{ y \mid g(y) = -y_1 - \frac{1}{4}y_2^2 - y_3^2 \geq 0 \right\}$$

Example

- Professor W is working at full capacity
- Professor W's output (on bdry) $y^0 = (-25, 8, 3)$
- What kind of reward scheme can support this?

$$p = -\frac{\partial g}{\partial y}(y^0) = (1, \frac{1}{2}y_2^0, 2y_3^0) = (1, 4, 6)$$

- How can you induce $(y_2^1, y_3^1) = (2, 2\sqrt{6}) \approx (2, 5)$

$$p = (1, \frac{1}{2}y_2^1, 2y_3^1) = (1, 1, 4\sqrt{6}) \approx (1, 1, 10)$$

Positive Prices (Free Disposal)

- Supporting Hyperplane theorem has economic meaning if prices are positive
 - Need another assumption
- **Free Disposal**
- For any feasible production plan $y \in \mathcal{Y}$ and any
- $\delta > 0$, the production plan $y - \delta$ is also feasible

Supporting Prices

- With free disposal, we can prove:

Proposition 1.1-3 (**Supporting Prices**)

- If y^0 is a boundary point of a convex set \mathcal{Y}
- And the free disposal assumption holds,
- Then, there exists a price vector $p > 0$ such
- that $p \cdot y \leq p \cdot y^0$ for all $y \in \mathcal{Y}$
- Moreover, if $y \in \mathcal{Y}$, then $p \cdot y^0 > 0$

Supporting Prices

- **Proof:** Supporting Hyperplane Theorem says:
- There is some $p \neq 0$ such that, for all $y \in \mathcal{Y}$,
- $p \cdot (y^0 - y) \geq 0$. Now need to show $p_i \geq 0$
- By free disposal, $y' = y^0 - \delta \in \mathcal{Y}$ for all $\delta > 0$
- Setting $\delta = (1, 0, \dots, 0)$, $p \cdot (y^0 - y') = p_1 \geq 0$
- Setting $\delta = (0, 1, 0, \dots)$, $p \cdot (y^0 - y') = p_2 \geq 0$
- ...
- Setting $\delta = (0, \dots, 0, 1)$, $p \cdot (y^0 - y') = p_n \geq 0$

Back to Publication Rewards

- Should NTU really pay NT\$300,000 per article published in Science or Nature?
 - Is the production set for Science/Nature convex?
- What would be a better incentive scheme to encourage publications in Science/Nature?
 - Efficient Wages (High Fixed Wages)?
 - Tenure?
 - Endowed Chair Professorships?

Back to Publication Rewards

- What are some tasks do you expect piece-rate incentives to work?
 - Sales
 - Real estate agents
- What about a fixed payment?
 - Secretaries and Office Staff
 - Store Clerk
- What about other incentives schemes?
 - That's for you to answer (in contract theory)!

Summary of 1.1

- Input = Negative Output
- Vector space of y
- Convexity (quasi-concavity) is the key for supporting prices (=linearization)
- What is a good incentive scheme to induce professor to publish in Science and Nature?
- Consumer = Producer
- Homework: Try Past Exam Questions