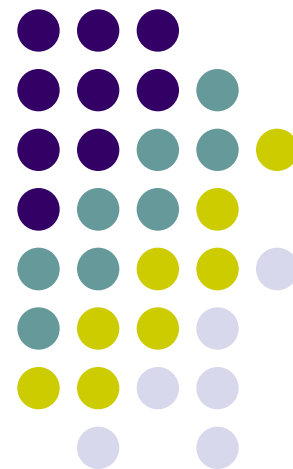


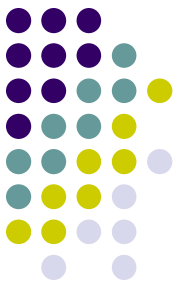
General Equilibrium for the Exchange Economy

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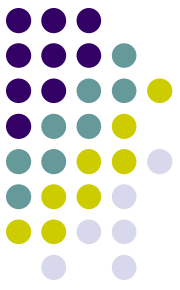
(Lecture 9, Micro Theory I)



What We Learned from the 2x2 Economy?

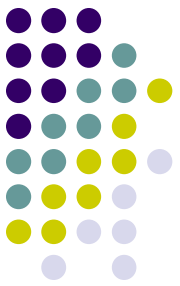


- Pareto Efficient Allocation (PEA)
 - Cannot make one better off without hurting others
- Walrasian Equilibrium (WE)
 - When Supply Meets Demand
 - Focus on Exchange Economy First
- 1st Welfare Theorem: WE is Efficient
- 2nd Welfare Theorem: Any PEA can be supported as a WE
- These also apply to the general case as well!



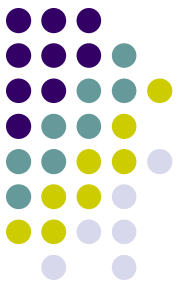
General Exchange Economy

- n Commodities: $1, 2, \dots, n$
- H Consumers: $h = 1, 2, \dots, H$
 - Consumption Set: $X^h \subset \mathbb{R}^n$
 - Endowment: $\omega^h = (\omega_1^h, \dots, \omega_n^h) \in X^h$
 - Consumption Vector: $x^h = (x_1^h, \dots, x_n^h) \in X^h$
 - Utility Function: $U^h(x^h) = U^h(x_1^h, \dots, x_n^h)$
 - Aggregate Consumption and Endowment:
$$x = \sum_{h=1}^H x^h \text{ and } \omega = \sum_{h=1}^H \omega^h$$
- Edgeworth Cube (Hyperbox)



Feasible Allocation

- A allocation is **feasible** if
- The sum of all consumers' demand **doesn't exceed** aggregate endowment: $x - \omega \leq 0$
- A feasible allocation \bar{x} is **Pareto efficient** if
- there is no other feasible allocation x that is
- **strictly preferred** by at least one: $U^i(x^i) > U^i(\bar{x}^i)$
- and is **weakly preferred** by all: $U^h(x^h) \geq U^h(\bar{x}^h)$



Walrasian Equilibrium

- Price-taking: Prices $p \geq 0$
- Consumers: $h=1, 2, \dots, H$
 - Endowment: $\omega^h = (\omega_1^h, \dots, \omega_n^h)$ $\omega = \sum_h \omega^h$
 - Wealth: $W^h = p \cdot \omega^h$
 - Budget Set: $\{x^h \in X^h \mid p \cdot x^h \leq W^h\}$
 - Consumption Set: $\bar{x}^h = (\bar{x}_1^h, \dots, \bar{x}_n^h) \in X^h$
- Most Preferred Consumption:
 $U^h(\bar{x}^h) \geq U^h(x^h)$ for all x^h such that $p \cdot x^h \leq W^h$
- Vector of Excess Demand: $\bar{e} = \bar{x} - \omega$

Definition:

Walrasian Equilibrium Prices



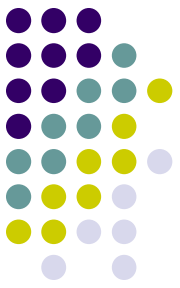
- The price vector $p \geq 0$ is a **Walrasian Equilibrium price vector** if
 - there is no market in excess demand ($\bar{e} \leq 0$),
 - and $p_j = 0$ for any market that is in excess supply ($\bar{e}_j < 0$).
-
- We are now ready to state and prove the “Adam Smith Theorem” (WE \rightarrow PEA)...

Proposition 3.2-1: First Welfare Theorem



- If preferences of each consumer satisfies LNS, then the Walrasian Equilibrium allocation is Pareto efficient.
- Proof:
 1. Since $U^h(x^h) > U^h(\bar{x}^h) \Rightarrow p \cdot x^h > p \cdot \omega^h$
 2. By LNS, $U^h(x^h) \geq U^h(\bar{x}^h) \Rightarrow p \cdot x^h \geq p \cdot \omega^h$
 3. Then, $\sum_h (p \cdot x^h - p \cdot \omega^h) = p \cdot (x - \omega) > 0$
- Which is not feasible ($x - \omega > 0$), since $p \geq 0$

First Welfare Theorem: WE \rightarrow PE



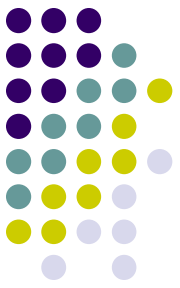
1. Why $U^h(x^h) > U^h(\bar{x}^h) \Rightarrow p \cdot x^h > p \cdot \omega^h$?

$$\bar{x}^h \text{ solves } \max_{x^h} \{U^h(x^h) \mid p \cdot x^h \leq p \cdot \omega^h\}$$

2. Why $U^h(x^h) \geq U^h(\bar{x}^h) \Rightarrow p \cdot x^h \geq p \cdot \omega^h$?

- Suppose not, then $p \cdot x^h < p \cdot \bar{x}^h$
- All bundles in sufficiently small neighborhood of x^h is in budget set $\{x^h \in X^h \mid p \cdot x^h \leq W^h\}$
- LNS requires a \hat{x}^h in this neighborhood to have $U^h(\hat{x}^h) > U^h(x^h)$, a contradiction.

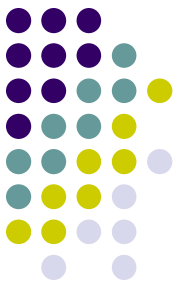
Lemma 3.2-2: Quasi-concavity of V



- If $U^h, h = 1, \dots, H$ is quasi-concave,
- Then so is the indirect utility function

$$V^i(x) = \max_{x^h} \left\{ U^i(x^i) \left| \sum_{h=1}^H x^h \leq x, \right. \right. \\ \left. \left. U^h(x^h) \geq U^h(\hat{x}^h), h \neq i \right\}$$

Lemma 3.2-2: Quasi-concavity of V



- Proof: Consider $V^i(b) \geq V^i(a)$, for any $c = (1 - \lambda)a + \lambda b$, need to show $V^i(c) \geq V^i(a)$

Assume $\{a^h\}_{h=1}^H$ solves $V^i(a)$,

$\{b^h\}_{h=1}^H$ solves $V^i(b)$,

$\{c^h\}_{h=1}^H$ is feasible since $c^h = (1 - \lambda)a^h + \lambda b^h$

$$\Rightarrow V^i(c) \geq U^i(c^i)$$

Now we only need to prove $U^i(c^i) \geq V^i(a)$.

Lemma 3.2-2: Quasi-concavity of V



- Since $\{a^h\}_{h=1}^H$ solves $V^i(a)$,
 $\{b^h\}_{h=1}^H$ solves $V^i(b)$,

$$U^i(a^i) = V^i(a) \text{ and } U^i(b^i) = V^i(b) \geq V^i(a)$$

$$\Rightarrow U^i(c^i) \geq V^i(a) \text{ by quasi-concavity of } U^i$$

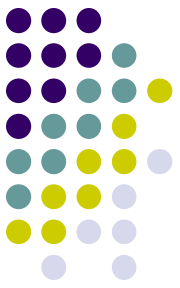
$$\Rightarrow V^i(c) \geq U^i(c^i) \geq V^i(a)$$

- **Note:** (By quasi-concavity of U^h)

$$U^h(a^h) \geq U^h(\hat{x}^h) \text{ for all } h \neq i \Rightarrow U^h(c^h) \geq U^h(\hat{x}^h)$$

$$U^h(b^h) \geq U^h(\hat{x}^h) \text{ for all } h \neq i$$

Proposition 3.2-3: Second Welfare Theorem



- Suppose $X^h = \mathbb{R}_+^n$, and utility functions $U^h(\cdot)$
- continuous, quasi-concave, strictly monotonic.
- If $\{\hat{x}^h\}_{h=1}^H$ is Pareto efficient, then there exist a price vector $p \geq 0$ such that

$$U^h(x^h) > U^h(\hat{x}^h) \Rightarrow p \cdot x^h > p \cdot \hat{x}^h$$

- Proof:

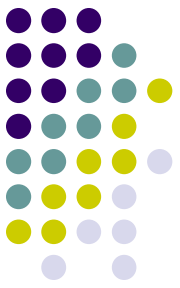
Proposition 3.2-3: Second Welfare Theorem



- Proof: Assume nobody has zero allocation
 - Relaxing this is easily done...
- By Lemma 3.2-2, $V^i(x)$ is quasi-concave
- $V^i(x)$ is strictly increasing since $U^i(\cdot)$ is also
 - (and any increment could be given to consumer i)
- Since $\{\hat{x}^h\}_{h=1}^H$ is Pareto efficient, $V^i(\omega) = U^i(\hat{x}^i)$
- Since $U^i(\cdot)$ is strictly increasing,

$$\sum_{h=1}^H \hat{x}^h = \omega$$

Proposition 3.2-3: Second Welfare Theorem



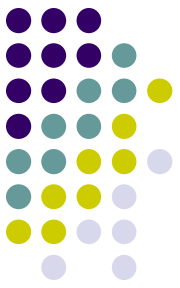
- Proof (Continued):
- Since ω is on the boundary of $\{x | V^i(x) \geq V^i(\omega)\}$
- By the Supporting Hyperplane Theorem, there exists a vector $p \neq 0$ such that

$$V^i(x) > V^i(\omega) \Rightarrow p \cdot x > p \cdot \omega$$
$$\text{and } V^i(x) \geq V^i(\omega) \Rightarrow p \cdot x \geq p \cdot \omega$$

- Claim: $p > 0$, then,

$$U^h(x^h) \geq U^h(\hat{x}^h) \Rightarrow p \cdot \sum_{h=1}^H x^h \geq p \cdot \omega = p \cdot \sum_{h=1}^H \hat{x}^h$$

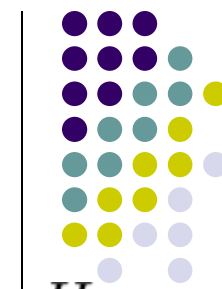
Proposition 3.2-3: Second Welfare Theorem



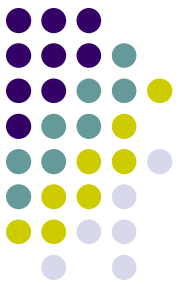
- Proof (Continued):
- Why $p > 0$? If not, define $\delta = (\delta_1, \dots, \delta_n) > 0$ such that $\delta_j > 0$ iff $p_j < 0$ (others = 0)
- Then, $V^i(\omega + \delta) > V^i(\omega)$ and $p \cdot (\omega + \delta) < p \cdot \omega$
- Contradicting (result from the Supporting Hyperplane Theorem)

$$U^h(x^h) \geq U^h(\hat{x}^h) \Rightarrow p \cdot \sum_{h=1}^H x^h \geq p \cdot \omega$$

Proposition 3.2-3: Second Welfare Theorem



- Since $U^h(x^h) \geq U^h(\hat{x}^h) \Rightarrow p \cdot \sum_{h=1}^H x^h \geq p \cdot \sum_{h=1}^H \hat{x}^h$
 - Set $x^k = \hat{x}^k$, $k \neq h$, then for consumer h
$$U^h(x^h) \geq U^h(\hat{x}^h) \Rightarrow p \cdot x^h \geq p \cdot \hat{x}^h$$
 - **Need to show strict inequality implies strict...**
 - If not, then $U^h(x^h) > U^h(\hat{x}^h) \Rightarrow p \cdot x^h = p \cdot \hat{x}^h$
 - Hence, $p \cdot \lambda x^h < p \cdot \hat{x}^h$ for all $\lambda \in (0, 1)$
- U^h continuous $\Rightarrow U^h(\lambda x^h) > U^h(\hat{x}^h)$ for large λ
- **Contradiction!**



Summary of 3.2

- Pareto Efficiency:
 - Cannot make one better off without hurting others
- Walrasian Equilibrium: market clearing prices
- Welfare Theorems:
 - First: Walrasian Equilibrium is Pareto Efficient
 - Second: Pareto Efficient allocations can be supported as Walrasian Equilibria (with transfer)
- Homework: Riley - 3.2-1~3
- J/R – 5.19, 5.27