General Equilibrium for the Exchange Economy

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(Lecture 9, Micro Theory I)

What We Learned from the 2x2 Economy?

- Pareto Efficient Allocation (PEA)
  - Cannot make one better off without hurting others
- Walrasian Equilibrium (WE)
  - When Supply Meets Demand
  - Focus on Exchange Economy First
- 1st Welfare Theorem: WE is Efficient
- 2nd Welfare Theorem: Any PEA can be supported as a WE
- These also apply to the general case as well!

General Exchange Economy

- *n* Commodities: *1, 2, ..., n*
- *H* Consumers: $h = 1, 2, \cdots, H$
  - Consumption Set: $X^h \subset \mathbb{R}^n$
  - Endowment: $\omega^h = (\omega_1^h, \cdots, \omega_n^h) \in X^h$
  - Consumption Vector: $x^h = (x_1^h, \cdots, x_n^h) \in X^h$
  - Utility Function: $U^h(x^h) = U^h(x_1^h, \cdots, x_n^h)$
  - Aggregate Consumption and Endowment:

$$x = \sum_{h=1}^{H} x^h \text{ and } \omega = \sum_{h=1}^{H} \omega^h$$

- Edgeworth Cube (Hyperbox)

Feasible Allocation

- A allocation is feasible if
- The sum of all consumers' demand doesn't exceed aggregate endowment: $x - \omega \leq 0$
- A feasible allocation $\overline{x}$ is Pareto efficient if
- there is no other feasible allocation x that is
- strictly preferred by at least one: $U^i(x^i) > U^i(\overline{x}^i)$
- and is weakly preferred by all: $U^h(x^h) \geq U^h(\overline{x}^h)$

Walrasian Equilibrium

- Price-taking: $\text{Prices } p \geq 0$
- Consumers: *h=1, 2, ..., H*
  - Endowment: $\omega^h = (\omega_1^h, \cdots, \omega_n^h) \qquad \omega = \sum_h \omega^h$
  - Wealth: $W^h = p \cdot \omega^h$
  - Budget Set: $\{x^h \in X^h | p \cdot x^h \leq W^h\}$
  - Consumption Set: $\overline{x}^h = (\overline{x}_1^h, \cdots, \overline{x}_n^h) \in X^h$
- Most Preferred Consumption:

$$U^h(\overline{x}^h) \geq U^h(x^h) \text{ for all } x^h \text{ such that } p \cdot x^h \leq W^h$$

- Vector of Excess Demand: $\overline{e} = \overline{x} - \omega$

Definition: Walrasian Equilibrium Prices

- The price vector $p \geq 0$ is a Walrasian Equilibrium price vector if
- there is no market in excess demand ($\overline{e} \leq 0$),
- and $p_j = 0$ for any market that is in excess supply ($\overline{e}_j < 0$).

- We are now ready to state and prove the "Adam Smith Theorem" (WE ➔ PEA)…

Proposition 3.2-1: First Welfare Theorem

- If preferences of each consumer satisfies LNS, then the Walrasian Equilibrium allocation is Pareto efficient.
- Proof:

1. Since $U^h(x^h) > U^h(\overline{x}^h) \Rightarrow p \cdot x^h > p \cdot \omega^h$
2. By LNS, $U^h(x^h) \geq U^h(\overline{x}^h) \Rightarrow p \cdot x^h \geq p \cdot \omega^h$
3. Then, $\sum_h \left(p \cdot x^h - p \cdot \omega^h\right) = p \cdot (x - \omega) > 0$

- Which is not feasible $(x - \omega > 0)$, since $p \geq 0$

First Welfare Theorem: WE → PE

1. Why $U^h(x^h) > U^h(\overline{x}^h) \Rightarrow p \cdot x^h > p \cdot \omega^h$?

$$\overline{x}^h \text{ solves } \max_{x^h} \{U^h(x^h) | p \cdot x^h \leq p \cdot \omega^h\}$$

2. Why $U^h(x^h) \geq U^h(\overline{x}^h) \Rightarrow p \cdot x^h \geq p \cdot \omega^h$?

- Suppose not, then $p \cdot x^h < p \cdot \overline{x}^h$
- All bundles in sufficiently small neighborhood of x^h is in budget set $\{x^h \in X^h | p \cdot x^h \leq W^h\}$
- LNS requires a $\hat{x}^h$ in this neighborhood to have $U^h(\hat{x}^h) > U^h(x^h)$, a contradiction.

Lemma 3.2-2: Quasi-concavity of *V*

- If $U^h, h = 1, \cdots, H$ is quasi-concave,
- Then so is the indirect utility function

$$V^i(x) = \max_{x^h} \left\{ U^i(x^i) \middle| \sum_{h=1}^{H} x^h \leq x, \right.$$

$$\left. U^h(x^h) \geq U^h(\hat{x}^h), h \neq i \right\}$$

Lemma 3.2-2: Quasi-concavity of *V*

- Proof: Consider $V^i(b) \geq V^i(a)$, for any $c = (1-\lambda)a + \lambda b$, need to show $V^i(c) \geq V^i(a)$

$$\text{Assume } \{a^h\}_{h=1}^H \text{ solves } V^i(a),$$
$$\{b^h\}_{h=1}^H \text{ solves } V^i(b),$$
$$\{c^h\}_{h=1}^H \text{ is feasible since } c^h = (1-\lambda)a^h + \lambda b^h$$
$$\Rightarrow V^i(c) \geq U^i(c^i)$$

$$\text{Now we only need to prove } U^i(c^i) \geq V^i(a).$$

Lemma 3.2-2: Quasi-concavity of *V*

- Since $\{a^h\}_{h=1}^{H}$ solves $V^i(a)$,

  $\{b^h\}_{h=1}^{H}$ solves $V^i(b)$,

$$U^i(a^i) = V^i(a) \text{ and } U^i(b^i) = V^i(b) \geq V^i(a)$$

$$\Rightarrow U^i(c^i) \geq V^i(a) \text{ by quasi-concavity of } U^i$$

$$\Rightarrow V^i(c) \geq U^i(c^i) \geq V^i(a)$$

- Note: (By quasi-concavity of U^h)

$$\left.\begin{array}{l} U^h(a^h) \geq U^h(\hat{x}^h) \text{ for all } h \neq i \\ U^h(b^h) \geq U^h(\hat{x}^h) \text{ for all } h \neq i \end{array}\right. \Rightarrow U^h(c^h) \geq U^h(\hat{x}^h)$$

Proposition 3.2-3: Second Welfare Theorem

- Suppose $X^h = \mathbb{R}^n_+$, and utility functions $U^h(\cdot)$
- continuous, quasi-concave, strictly monotonic.
- If $\{\hat{x}^h\}_{h=1}^{H}$ is Pareto efficient, then there exist a price vector $p \geq 0$ such that

$$U^h(x^h) > U^h(\hat{x}^h) \Rightarrow p \cdot x^h > p \cdot \hat{x}^h$$

- Proof:

Proposition 3.2-3: Second Welfare Theorem

- Proof: Assume nobody has zero allocation
  - Relaxing this is easily done…
- By Lemma 3.2-2, $V^i(x)$ is quasi-concave
- $V^i(x)$ is strictly increasing since $U^i(\cdot)$ is also
  - (and any increment could be given to consumer i)
- Since $\{\hat{x}^h\}_{h=1}^H$ is Pareto efficient, $V^i(\omega) = U^i(\hat{x}^i)$
- Since $U^i(\cdot)$ is strictly increasing,

$$\sum_{h=1}^{H} \hat{x}^h = \omega$$

Proposition 3.2-3: Second Welfare Theorem

- Proof (Continued):
- Since ω is on the boundary of $\{x|V^i(x) \geq V^i(\omega)\}$
- By the Supporting Hyperplane Theorem, there exists a vector $p \neq 0$ such that

$$V^i(x) > V^i(\omega) \Rightarrow p \cdot x > p \cdot \omega$$
$$\text{and } V^i(x) \geq V^i(\omega) \Rightarrow p \cdot x \geq p \cdot \omega$$

- Claim: $p > 0$, then,

$$U^h(x^h) \geq U^h(\hat{x}^h) \Rightarrow p \cdot \sum_{h=1}^{H} x^h \geq p \cdot \omega = p \cdot \sum_{h=1}^{H} \hat{x}^h$$

Proposition 3.2-3: Second Welfare Theorem

- Proof (Continued):
- Why $p > 0$? If not, define $\delta = (\delta_1, \cdots, \delta_n) > 0$ such that $\delta_j > 0$ iff $p_j < 0$ (others = 0)
- Then, $V^i(\omega + \delta) > V^i(\omega)$ and $p \cdot (\omega + \delta) < p \cdot \omega$
- Contradicting (result from the Surporting Hyperplane Theorem)

$$U^h(x^h) \geq U^h(\hat{x}^h) \Rightarrow p \cdot \sum_{h=1}^{H} x^h \geq p \cdot \omega$$

Proposition 3.2-3: Second Welfare Theorem

- Since $U^h(x^h) \geq U^h(\hat{x}^h) \Rightarrow p \cdot \sum_{h=1}^{H} x^h \geq p \cdot \sum_{h=1}^{H} \hat{x}^h$
- Set $x^k = \hat{x}^k,\ k \neq h$, then for consumer *h*

$$U^h(x^h) \geq U^h(\hat{x}^h) \Rightarrow p \cdot x^h \geq p \cdot \hat{x}^h$$

- Need to show strict inequality implies strict…
- If not, then $U^h(x^h) > U^h(\hat{x}^h) \Rightarrow p \cdot x^h = p \cdot \hat{x}^h$
- Hence, $p \cdot \lambda x^h < p \cdot \hat{x}^h$ for all $\lambda \in (0,1)$

U^h continuous $\Rightarrow U^h(\lambda x^h) > U^h(\hat{x}^h)$ for large λ

- Contradiction!

Summary of 3.2

- Pareto Efficiency:
  - Cannot make one better off without hurting others
- Walrasian Equilibrium: market clearing prices
- Welfare Theorems:
  - First: Walrasian Equilibrium is Pareto Efficient
  - Second: Pareto Efficient allocations can be supported as Walrasian Equilibria (with transfer)
- Homework: Riley - 3.2-1~3
- J/R – 5.19, 5.27