Supporting Prices and Convexity

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(Lecture 8, Micro Theory I)

Overview of Chapter 1

- Theory of Constrained Maximization
  - Why should we care about this?
- What is Economics?
- Economics is the study of how society manages its scarce resources (Mankiw, Ch.1)
  - “Economics is the science which studies human behavior as a relationship between given ends and scarce means which have alternative uses.” (Lionel Robbins, 1932)

Overview of Chapter 1

- Other Historical Accounts:
  - Economics is the "study of how societies use scarce resources to produce valuable commodities and distribute them among different people." (Paul A. Samuelson, 1948)
- My View: Economics is a study of institutions and human behavior (reactions to institutions)
- Either way, constrained maximization is key…

Overview of Chapter 1

- Tools Introduced in Chapter 1:

1. Supporting Hyperplanes (and Convexity)
2. First Order Conditions (Kuhn-Tucker)
3. Envelope Theorem

- But why do I need to know the math?
  - Along the way, please let me know if you expect to use these tools in the future (in work)…

Publication Reward Problem

- Example: How should NTU reward its professors to publish journal articles?
  - Should NTU pay, say, NT$300,000 per article published in Science or Nature?
- Well, it depends…
- Peek the answer ahead:
  - Yes, if the production set is convex.
  - No, if, for example, there is initial increasing returns to scale.

Supporting Prices

- More generally, can prices and profit maximization provide appropriate incentives for all efficient production plans?
  - Is there a price vector that supports each efficient production plan?
- (Yes, but when?)
- Need some definitions first…

Production Plant

- A production facility can produce n output $q=(q_1,\ldots,q_n)$ using up to m input $z=(z_1,\ldots,z_m)$
- Production Plan $y=(-z_1,\ldots,-z_m,q_1,\ldots,q_n)$
- Price vector $p=(p_1,\ldots,p_{m+n})$
- Profit $\Pi=\underbrace{\sum_{i=m+1}^{m+n} p_i y_i}_{\text{total revenue}}-\underbrace{\sum_{i=1}^{m} p_i z_i}_{\text{total cost}}=p\cdot y$

Production Plan

- Production Set Υ

  =Set of Feasible Production Plan

- $\bar{y}$ is production efficient (=non-wasteful) if

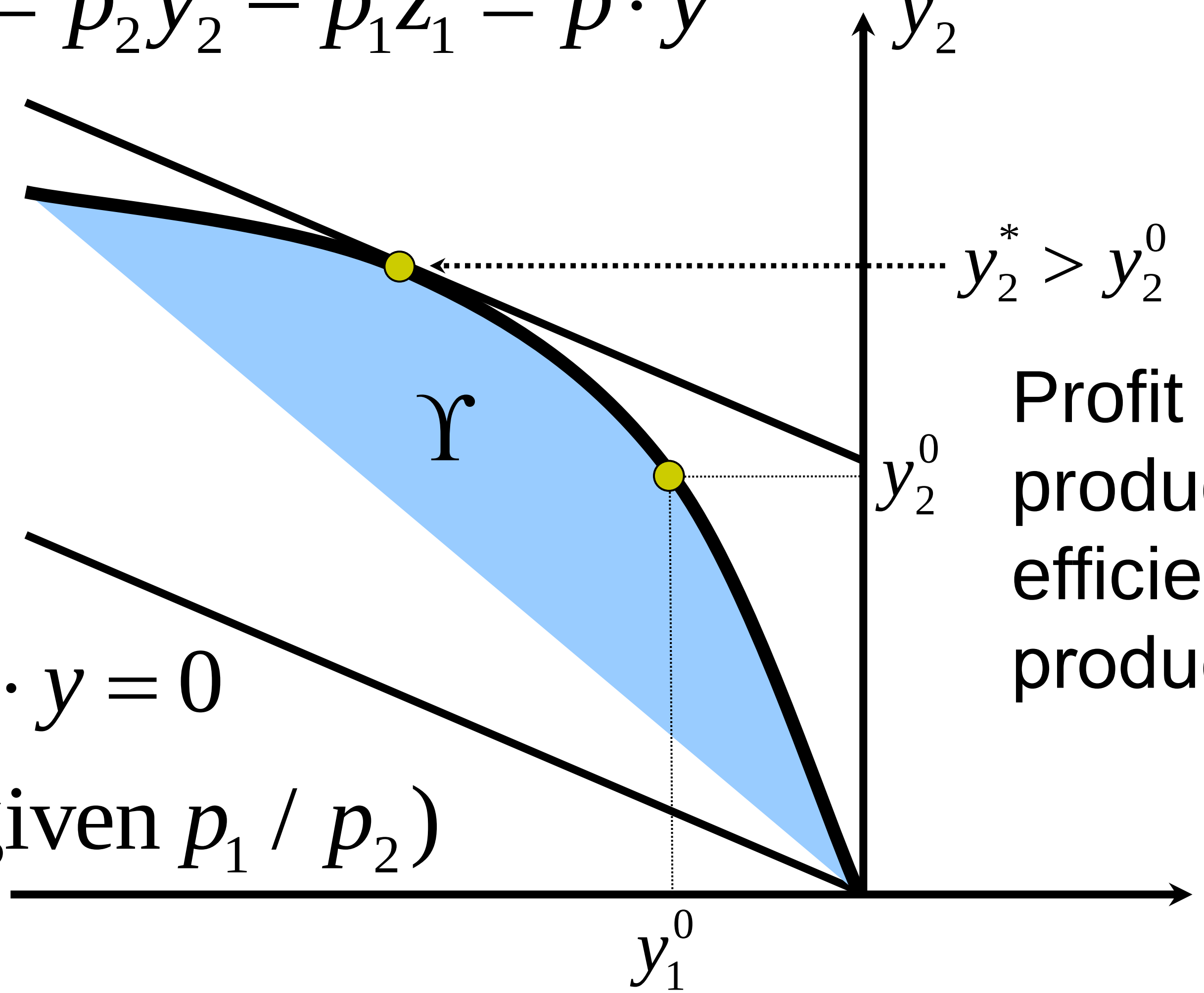
  There is no $y \in \Upsilon$ such that $y > \bar{y}$

  - Note: $y \geq \bar{y}$ if $y_j \geq \bar{y}_j$ for all j

    $y > \bar{y}$ if inequality is strict for some j

Can Prices Support an Efficient Production Plan?

$$\Pi = p_2 y_2 - p_1 z_1 = p \cdot y$$



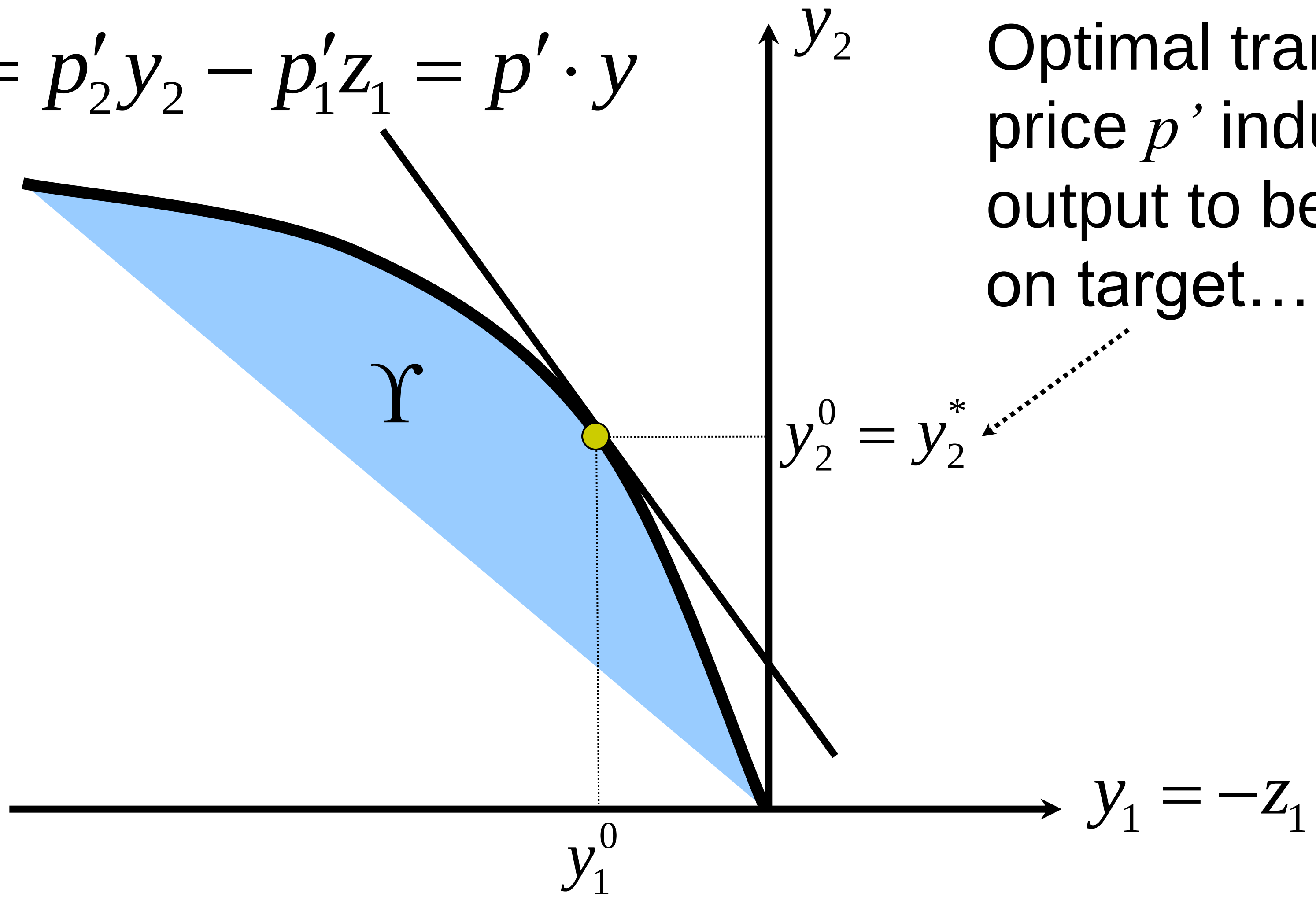
$$p \cdot y = 0$$

$$(\text{given } p_1 / p_2)$$

Profit maximizing production plan is efficient, but over-produces…

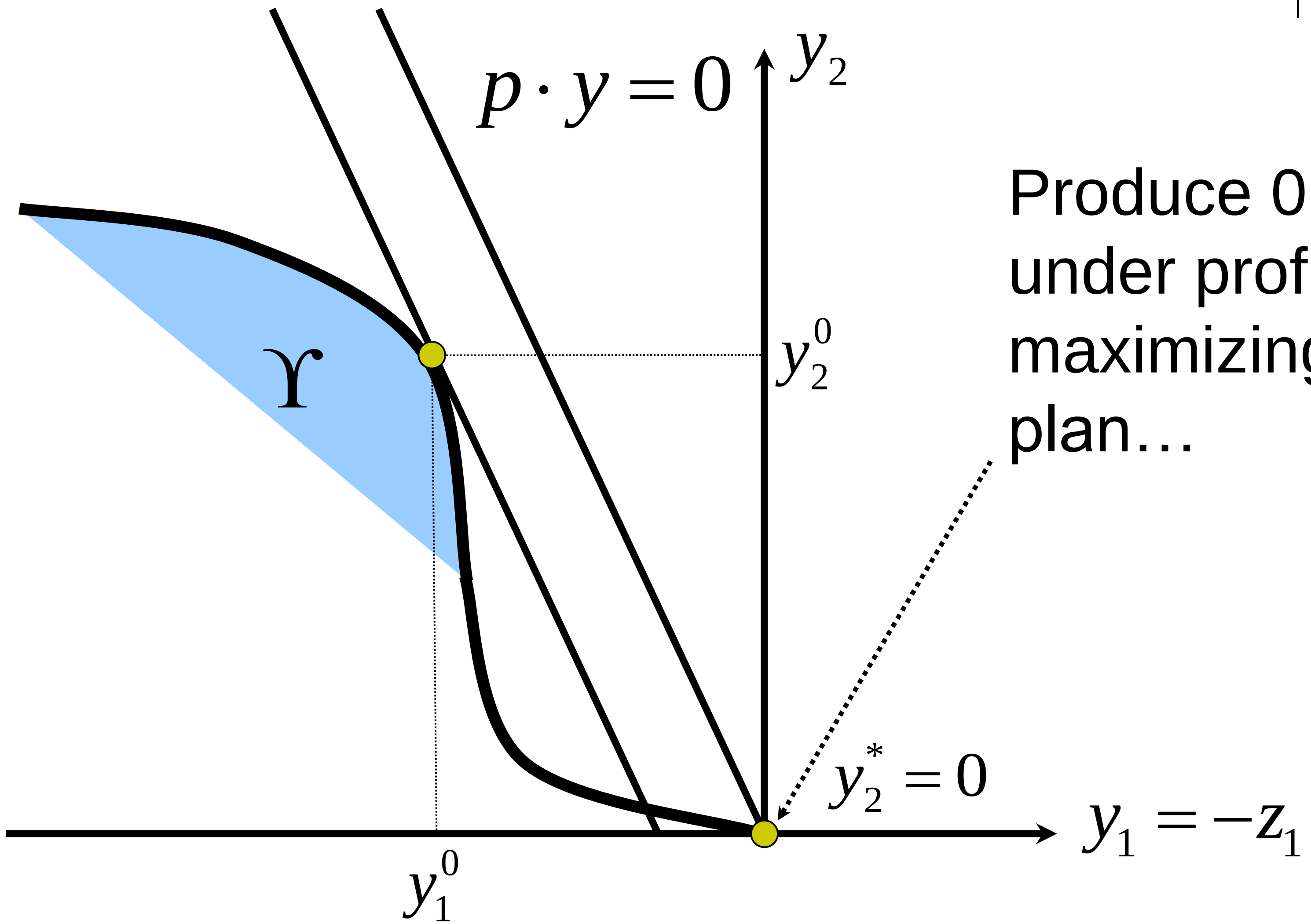
If That Was Too High, Let's Lower the Transfer Price…

$$\Pi = p_2' y_2 - p_1' z_1 = p' \cdot y$$



Optimal transfer price p' induces output to be right on target…

Will this Always Work?



Produce 0 under profit maximizing plan…

What Made It Fail?

- The last production set was NOT convex.

- S is convex iff $\text{for all } \mathrm{y}^0, \mathrm{y}^1 \in S,$

$$\mathrm{y}^\lambda = \lambda \cdot \mathrm{y}^0 + (1 - \lambda) \cdot \mathrm{y}^1 \in S$$

- Is it true that we can use prices to guide production decisions as long as production sets are convex?

Supporting Hyperplane Theorem

- Proposition 1.1-1:

Suppose Υ is closed and convex,

and y^0 lies on the boundary of Υ.

Then there exists $p \neq 0$, such that

(i) for all $y \in \Upsilon$, $p \cdot y \leq p \cdot y^0$, and

(ii) for all $y \in \text{int}\Upsilon$, $p \cdot y < p \cdot y^0$.

Special Case of Supporting Hyperplane Theorem

- "Proposition" 1.1- "1.5":

The production set Υ is the upper contour set

$\{ y \mid h(y) \geq h(y^0) \}$ of h, quasi-concave, differentiab

Suppose the gradient vector of h is non-zero at y^0.

$$\text{Define } p = -\frac{\partial h}{\partial y}(y^0),$$

then $p \cdot y \leq p \cdot y^0$ for all $y \in \Upsilon$.

Quasi-Concavity

- f is quasi-concave if the upper contour set of f set are convex. Equivalently, for any y^0, y^1 and convex combination

$$y^{\lambda} = \lambda \cdot y^0 + (1-\lambda) \cdot y^1,$$

$$f(y^{\lambda}) \geq \min\left\{ f(y^0), f(y^1) \right\}.$$

- Why is this useful?
  - Because we have…

Quasi-Concavity

- If f is differentiable and quasi-concave, then

$$f(\mathrm{y}) \geq f(\mathrm{y}^0) \text{ implies } \frac{\partial f}{\partial x}(y^0)\cdot(y - y^0) \geq 0$$

- This tells us how to calculate the supporting prices (under this special case)…

Example

- A professor has 25 units of "brain-power"
- Allocates z_1 units to produce TSSCI papers

Produce $y_1 = 2\sqrt{z_1}$ number of TSSCI papers

- Allocates z_2 units to produce SSCI papers

Produce $y_2 = \sqrt{z_2}$ number of SSCI papers

- Set of feasible output is

$$\Upsilon = \left\{ y \mid h(y) = 25 - \frac{1}{4} {y_1}^2 - {y_2}^2 \right\}$$

Example

- Professor W is working at full capacity

  Professor W's output $y^0 = (8,3)$ (on the boundary)

- What kind of reward scheme can support this?

$$p = -\frac{\partial h}{\partial y}(y^0) = \left(\frac{1}{2}y_1^0, 2y_2^0\right) = (4,6)$$

- How can university induce $y^1 = \left(2, 2\sqrt{6}\right) \approx (2,5)$?

$$p = -\frac{\partial h}{\partial y}(y^1) = \left(\frac{1}{2}y_1^1, 2y_2^1\right) = \left(1, 4\sqrt{6}\right) \approx (1,10)$$

Separating Hyperplane Theorem

- Proposition 1.1-2:

  Suppose S and T are convex sets

  with a common boundary point $s^0 = t^0$

  and no common interior points.

  Then there is some p such that,

  for all $s \in S$ and $t \in T$, $p \cdot s \leq p \cdot t$.

  (Inequality strict if either s or t is an interior.)

Separating Hyperplane Theorem

- Proof of Proposition 1.1-2:

Define $\Upsilon = S - T$, then $s^0 - t^0 = 0 \in \Upsilon$

If Υ is convex (verify this!!!), then...

Supporting Hyperplane Theorem says:

there is some $p \neq 0$ such that, for all $y \in \Upsilon$,

$p \cdot y \leq p \cdot (s^0 - t^0) = 0.$

Since $y = s - t$ for some $s \in S, t \in T$,

$p \cdot s \leq p \cdot t$ for all $s \in S, t \in T$.

Positive Prices (Free Disposal)

- Two hyperplane theorems have economic meaning if prices are positive
  - Need another assumption
- Free Disposal

For any feasible produciton plan $y \in \Upsilon$

and any $\delta > 0$, the production plan $y - \delta$

is also feasible.

Supporting Prices

- With free disposal, we can prove:
- Proposition 1.1-3:

If y^0 is a boundary point of a convex sets Υ and the free disposal assumption holds, then Then there exist a price vector $p \geq 0$ such that $p \cdot y \leq p \cdot y^0$ for all $y \in \Upsilon$.

(Moreover, if $0 \in \Upsilon$, then $p \cdot y^0 \geq 0$.)

Supporting Prices

- Proof of Proposition 1.1-3:
  Supporting Hyperplane Theorem says:

  there is some $p \neq 0$ such that, for all $y \in \Upsilon$,

  $p \cdot (y^0 - y) \geq 0$. Now need to show $p_i \geq 0$.

  By free disposal, $y' = y^0 - \delta \in \Upsilon$ for all $\delta > 0$.

  Setting $\delta = (1, 0, \ldots, 0)$, $p \cdot (y^0 - y') = p_1 \geq 0$.

  Setting $\delta = (0, 1, 0, \ldots, 0)$, $p \cdot (y^0 - y') = p_2 \geq 0$.

  ...Setting $\delta = (0, \ldots, 0, 1)$, $p \cdot (y^0 - y') = p_n \geq 0$.

Back to Publication Rewards

- Should NTU really pay NT$300,000 per article published in Science or Nature?
  - Is the production set for Science/Nature convex?
- What would be a better incentive scheme to encourage publications in Science/Nature?
  - Efficient Wages (High Fixed Wages)?
  - Tenure?
  - Endowed Chair Professorships?

Back to Publication Rewards

- What are some tasks do you expect piece-rate incentives to work?
  - Sales
  - Real estate agents
- What about a fixed payment?
  - Secretaries and Office Staff
  - Store Clerk
- What about other incentives schemes?
  - That’s for you to answer (in contract theory)!

Summary of 1.1

- Input = Negative Output
- Vector space of y
- Convexity (quasi-concavity) is the key for supporting prices (=linearization)
- What is a good incentive scheme to induce professor to publish in Science and Nature?
- Consumer = Producer
- Homework: Try Last Year's Exam questions