Budget Constrained Choice with Two Commodities

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(Lecture 4, Micro Theory I)

The Consumer Problem

- We have some powerful tools:
  - Constrained Maximization (Shadow Prices)
  - Envelope Theorem (Changing Environment)
- How can they help us understand behavior of a consumer?
  - Either “maximizing utility while facing a budget constraint”, or “minimizing cost while maintaining a certain welfare level”…

Key Problems to Consider

- Consumer Problem: How can consumer's Utility Maximization result in demand?
- Income Effect: How does an increase (or decrease) in income (budget) affect demand?
- Dual Problem: How is Minimizing Expenditure related to Maximizing Utility?
- Substitution Effect: How does an increase in commodity price affect compensated demand?
- Total Price Effect = S. E. + I. E.

Why do we care about this? An Example in Public Policy

- Taiwan's ministry of defense has to decide whether to buy more fighter jets, or more submarines given a tight budget
- How does the military rank each combination?
- How do they choose which combination to buy?
- How would a price change affect their decision?
- How would a boycott in defense budget affect their decision?

Continuous Demand Function

A Consumer with income I, facing prices p_1, p_2

$$\max_x \{U(x) | p \cdot x \leq I, x \in \mathbb{R}^2_+\}$$

- Assume:
- LNS (local non-satiation)
  - Consumer spends all his/her income
- $U(x)$ is continuous, strictly quasi-concave on $\mathbb{R}^2_+$
  - There is a unique solution $x^0 = x(p, I)$
- Then, by Proposition 2.2-1,

$x(p, I)$ must be continuous.

Stronger Convenience Assumptions for this Lecture

- Assume:
- $U(x)$ is continuously differentiable on $\mathbb{R}^2_+$
  - FOC is gradient vectors of utility (+ constraint)
- LNS-plus: $$\frac{\partial U}{\partial x}(x) > 0 \text{ for all } x \in \mathbb{R}^2_+$$
  - At least one commodity has MU > 0
- No corners: $$\lim_{x_j \to 0} \frac{\partial U}{\partial x_j} = \infty, j = 1, 2$$
  - Always wants to consume some of everything

Indifference Curve Analysis (Lagrangian Version)

A Consumer with income I, facing prices p_1, p_2

$$\max_x \left\{ U(x) | p \cdot x \leq I, x \in \mathbb{R}_+^2 \right\}$$

Lagrangian is $\mathfrak{L} = U + \lambda(I - p \cdot x)$

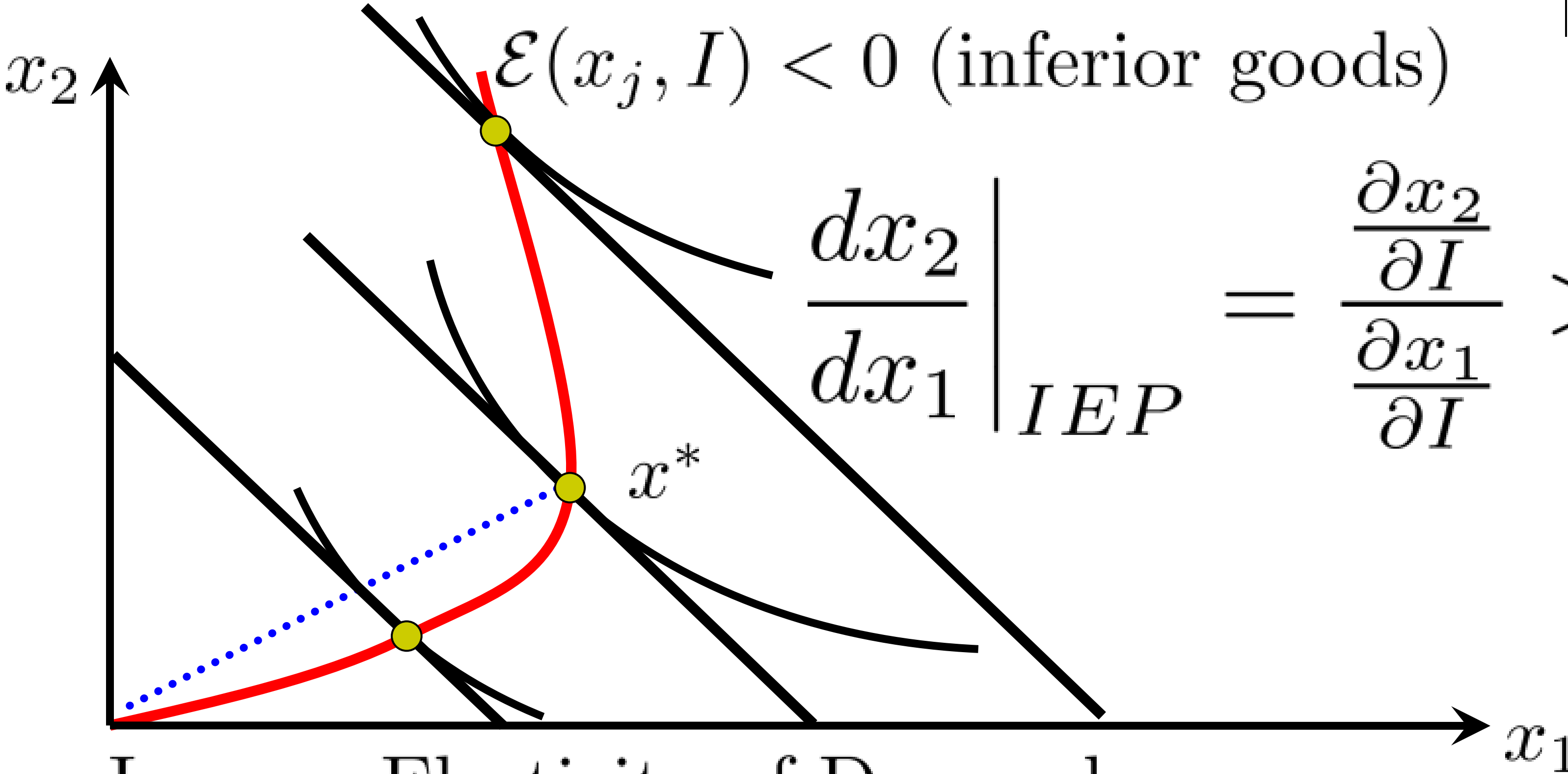
$$(FOC) \qquad \frac{\partial \mathfrak{L}}{\partial x_j} = \frac{\partial U}{\partial x_j}(x^*) - \lambda p_j = 0, j = 1, 2$$

$$\frac{\frac{\partial U}{\partial x_1}}{p_1} = \frac{\frac{\partial U}{\partial x_2}}{p_2} = \lambda$$

Meaning of FOC

1. Same marginal value for last dollar spent on each commodity
$$\frac{\frac{\partial U}{\partial x_1}}{p_1} = \frac{\frac{\partial U}{\partial x_2}}{p_2} = \lambda$$
   - Does Taiwan get same MU on fighter jets and submarines?
2. Indifference Curve tangent to Budget Line
$$MRS(x^*) = \frac{\frac{\partial U}{\partial x_1}}{\frac{\partial U}{\partial x_2}} = \frac{p_1}{p_2}$$

Income Effect



$$\left.\frac{dx_2}{dx_1}\right|_{IEP} = \frac{\frac{\partial x_2}{\partial I}}{\frac{\partial x_1}{\partial I}} > \frac{x_2^*}{x_1^*}$$

Income Elasticity of Demand

$$\mathcal{E}(x_j, I) = \frac{I}{x_j}\frac{\partial x_j}{\partial I} > 0 \text{ (normal goods)}$$

Income Effect

- Slope of IEP steeper than line joining 0 and x^*

$$\left.\frac{dx_2}{dx_1}\right|_{IEP} = \frac{\frac{\partial x_2}{\partial I}}{\frac{\partial x_1}{\partial I}} > \frac{x_2^*}{x_1^*}$$

- Or,

$$\mathcal{E}(x_2, I) = \frac{I}{x_2}\frac{\partial x_2}{\partial I} > \mathcal{E}(x_1, I) = \frac{I}{x_1}\frac{\partial x_1}{\partial I}$$

- Lemma 2.2-2: Expenditure share weighted income elasticity average = 1
- So, $\mathcal{E}(x_2, I) > 1 > \mathcal{E}(x_1, I)$

Three Examples

- Quasi-Linear Convex Preference

$$U(x) = v(x_1) + \alpha x_2$$

- Cobb-Douglas Preferences

$$U(x) = x_1^{\alpha_1} x_2^{\alpha_2}, \alpha_1, \alpha_2 > 0$$

- CES Utility Function

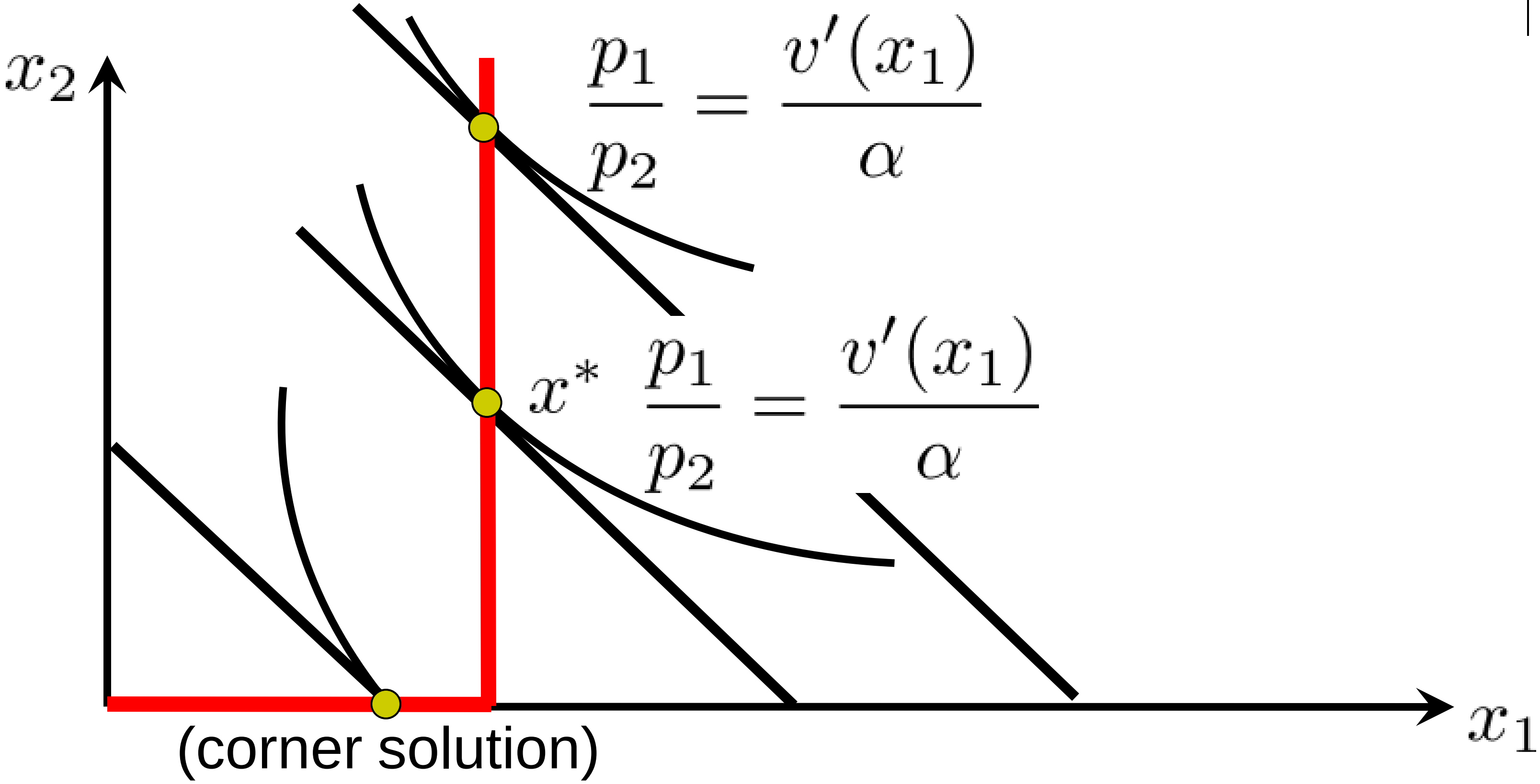
$$U(x) = \left(\alpha_1 x_1^{1-\frac{1}{\theta}} + \alpha_2 x_2^{1-\frac{1}{\theta}}\right)^{\frac{1}{1-\frac{1}{\theta}}}$$

Quasi-Linear Convex Utility

$$\max_x \left\{ U(x) = v(x_1) + \alpha x_2 \middle| p_1 x_1 + p_2 x_2 \le I, x \in \mathbb{R}^2_+ \right\}$$

- FOC: $$\frac{\frac{\partial U}{\partial x_1}}{p_1} = \frac{\frac{\partial U}{\partial x_2}}{p_2} = \frac{v'(x_1)}{p_1} = \frac{\alpha}{p_2} (= \lambda)$$
- Implication: $\frac{p_1}{p_2} = \frac{v'(x_1)}{\alpha}$ (MRS=price)
- Note that x_2 is irrelevant…
- What does this mean?

Income Effect



- Vertical Income Expansion Path…

Cobb-Douglas Preferences

$$\max_{x_1, x_2} U(x_1, x_2) = x_1^{\alpha_1} x_2^{\alpha_2}$$

$$\text{s.t. } P_{x_1} \cdot x_1 + P_{x_2} \cdot x_2 \leq I = P_{x_1} \cdot \omega_{x_1} + P_{x_2} \cdot \omega_{x_2}$$

$$\mathcal{L} = x_1^{\alpha_1} x_2^{\alpha_2} + \lambda \cdot [I - P_{x_1} \cdot x_1 - P_{x_2} \cdot x_2]$$

FOC: (for interior solutions)

$$\frac{\partial \mathcal{L}}{\partial x_1} = \alpha_1 \cdot \frac{x_2^{\alpha_2}}{x_1^{\alpha_2}} - \lambda \cdot P_{x_1} = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = \alpha_2 \cdot \frac{x_1^{\alpha_1}}{x_2^{\alpha_1}} - \lambda \cdot P_{x_2} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = I - P_{x_1} \cdot x_1 - P_{x_2} \cdot x_2 = 0$$

Cobb-Douglas Preferences

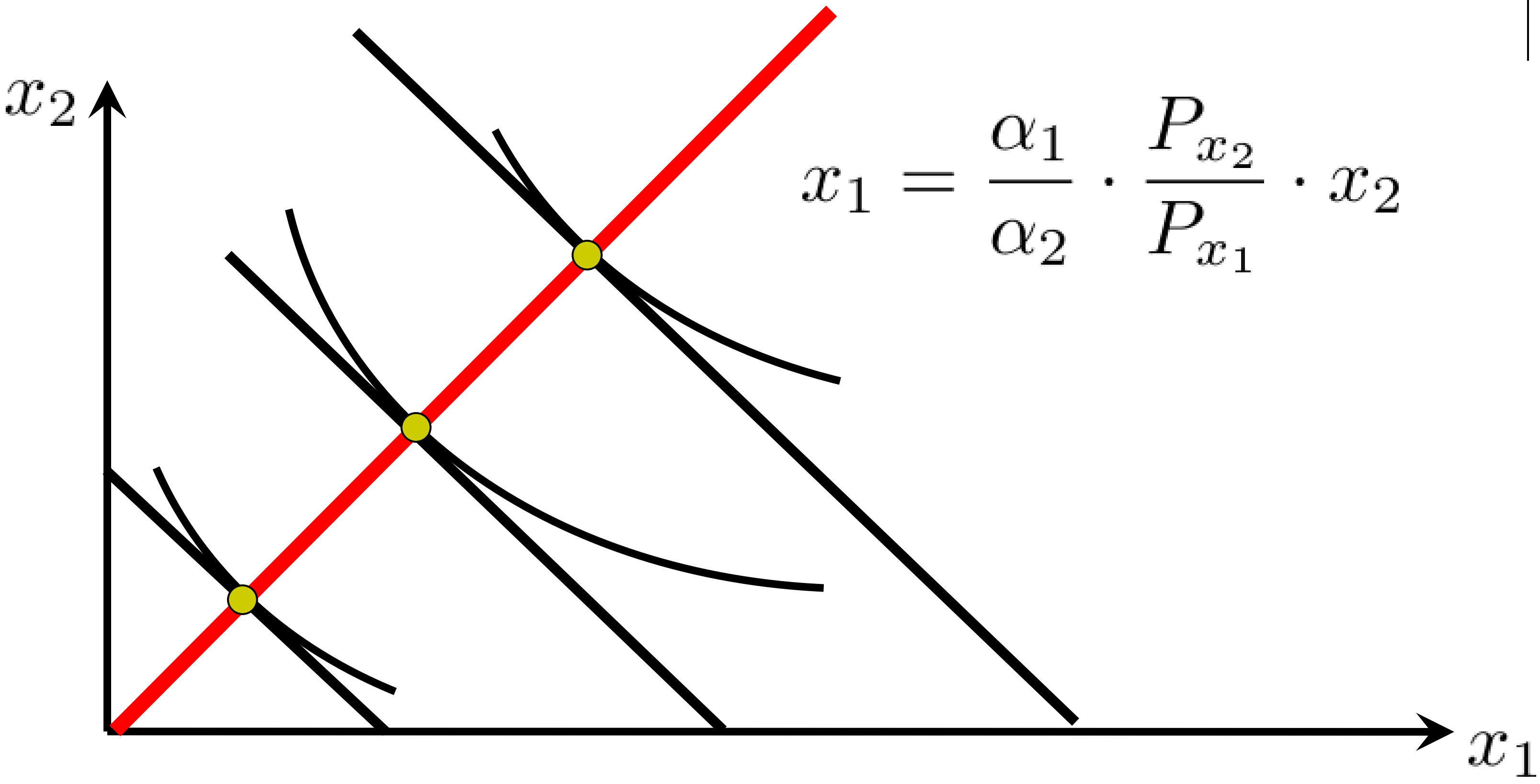
- Meaning of FOC: $MRS = \frac{P_{x_1}}{P_{x_2}}$

$$\frac{P_{x_1}}{P_{x_2}} = \frac{\alpha_1}{\alpha_2} \cdot \frac{x_2}{x_1} \qquad \Rightarrow x_1 = \frac{\alpha_1}{\alpha_2} \cdot \frac{P_{x_2}}{P_{x_1}} \cdot x_2$$

$$\Rightarrow I = P_{x_1} \cdot x_1 + P_{x_2} \cdot x_2 = \frac{\alpha_1 + \alpha_2}{\alpha_2} \cdot P_{x_2} \cdot x_2$$

$$\Rightarrow x_2^* = \frac{\alpha_2}{\alpha_1 + \alpha_2} \cdot \frac{I}{P_{x_2}}, \; x_1^* = \frac{\alpha_1}{\alpha_1 + \alpha_2} \cdot \frac{I}{P_{x_1}}$$

Income Effect



$$x_1 = \frac{\alpha_1}{\alpha_2} \cdot \frac{P_{x_2}}{P_{x_1}} \cdot x_2$$

- Linear Income Expansion Path…

CES Utility Function

$$U(x) = \left(\alpha_1 x_1^{1-\frac{1}{\theta}} + \alpha_2 x_2^{1-\frac{1}{\theta}}\right)^{\frac{1}{1-\frac{1}{\theta}}}$$

$$\mathcal{L} = \left(\alpha_1 x_1^{1-\frac{1}{\theta}} + \alpha_2 x_2^{1-\frac{1}{\theta}}\right)^{\frac{1}{1-\frac{1}{\theta}}} + \lambda \cdot \left[I^A - P_x \cdot x - P_y \cdot y\right]$$

- FOC: (for interior solutions)

$$\frac{\partial \mathcal{L}}{\partial x_1} = \alpha_1 x_1^{-\frac{1}{\theta}} \cdot \left(\alpha_1 x_1^{1-\frac{1}{\theta}} + \alpha_2 x_2^{1-\frac{1}{\theta}}\right)^{\frac{1}{\theta-1}} - \lambda \cdot P_{x_1} = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = \alpha_2 x_2^{-\frac{1}{\theta}} \cdot \left(\alpha_1 x_1^{1-\frac{1}{\theta}} + \alpha_2 x_2^{1-\frac{1}{\theta}}\right)^{\frac{1}{\theta-1}} - \lambda \cdot P_{x_2} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = I - P_{x_1} \cdot x_1 - P_{x_2} \cdot x_2 = 0$$

CES Utility Function

$$\frac{P_{x_1}}{P_{x_2}} = \frac{\alpha_1}{\alpha_2} \cdot \left(\frac{x_2}{x_1}\right)^{\frac{1}{\theta}} \Rightarrow x_1 = \left(\frac{\alpha_1}{\alpha_2} \cdot \frac{P_{x_2}}{P_{x_1}}\right)^{\theta} \cdot x_2$$

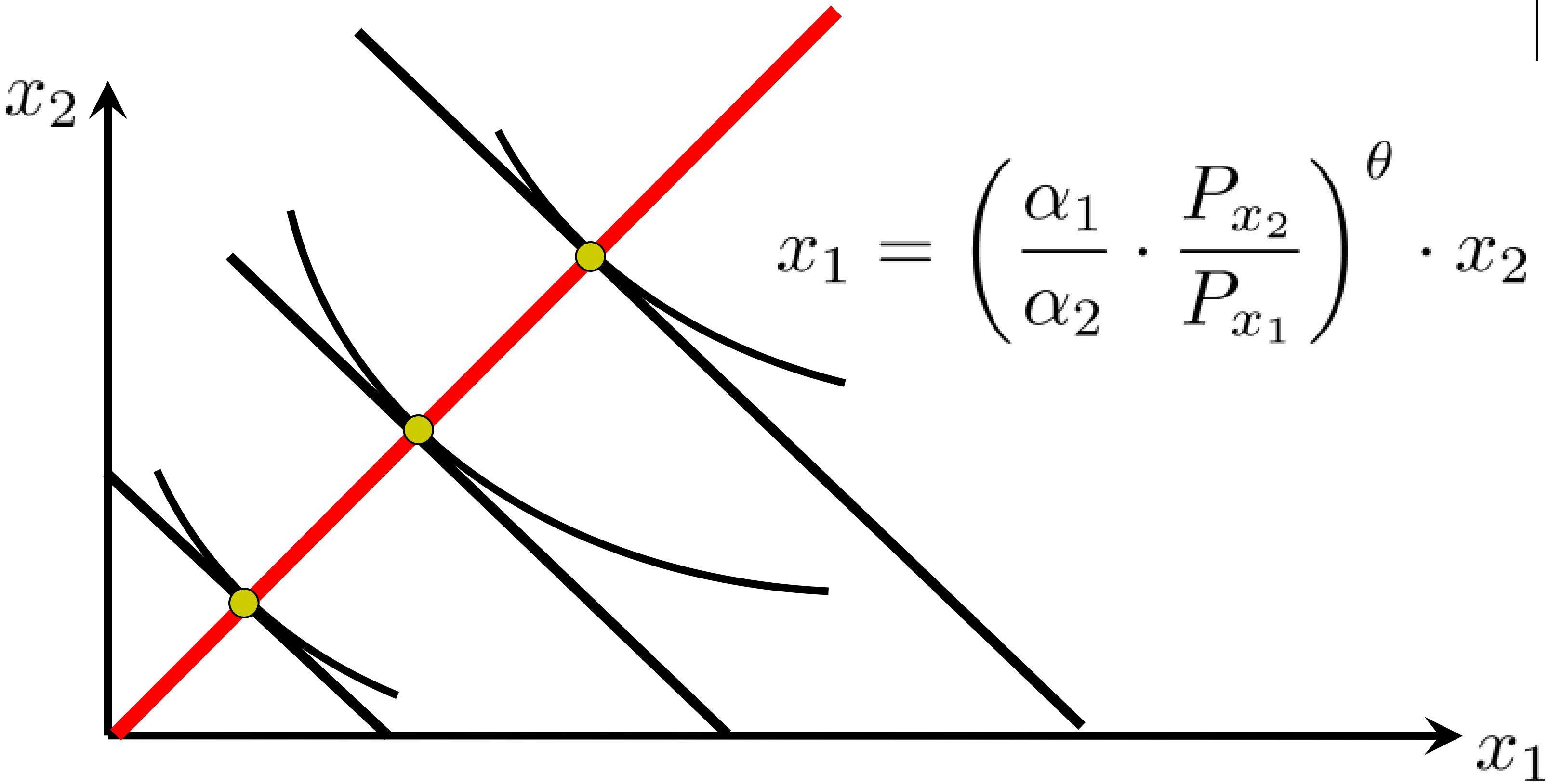
$$\Rightarrow I = P_{x_1} \cdot x_1 + P_{x_2} \cdot x_2$$

$$= \left[\left(\frac{\alpha_1}{\alpha_2}\right)^{\theta} \cdot \left(\frac{P_{x_2}}{P_{x_1}}\right)^{\theta-1}\right] \cdot P_{x_2} \cdot x_2$$

$$\Rightarrow x_2^* = \frac{\alpha_2^{\theta} P_{x_1}^{\theta-1}}{\alpha_1^{\theta} P_{x_2}^{\theta-1} + \alpha_2^{\theta} P_{x_1}^{\theta-1}} \cdot \frac{I}{P_{x_2}},$$

$$x_1^* = \frac{\alpha_1^{\theta} P_{x_1}^{\theta-1}}{\alpha_1^{\theta} P_{x_2}^{\theta-1} + \alpha_2^{\theta} P_{x_1}^{\theta-1}} \cdot \frac{I}{P_{x_1}}$$

Income Effect



$$x_1 = \left(\frac{\alpha_1}{\alpha_2} \cdot \frac{P_{x_2}}{P_{x_1}} \right)^{\theta} \cdot x_2$$

- Linear Income Expansion Path…
- Cobb-Douglas is a special case of CES!

Dual Problem: Minimizing Expenditure

- Consider the least costly way to achieve $\overline{U}$

$$M(p,\overline{U}) = \min_x \{p \cdot x | U(x) \geq \overline{U}\}$$

- How can you solve this?

$$\mathfrak{L} = -p \cdot x + \lambda(U(x) - \overline{U})$$

$$(FOC) \quad \frac{\partial \mathfrak{L}}{\partial x_j} = -p_j + \lambda \frac{\partial U}{\partial x_j}(x^*) = 0, j = 1,2$$

$$\frac{p_1}{\frac{\partial U}{\partial x_1}} = \frac{p_2}{\frac{\partial U}{\partial x_2}} = \lambda \Rightarrow \text{Solve for } x^c(p,\overline{U})$$

Dual Problem: Minimizing Expenditure

- We can also use it's "sister" (dual) problem:

$$\max_x \{U(x) | p \cdot x \leq I\}$$

- Note that, for $x(p, I)$ solving this problem,
- $U(x(p, I))$ is strictly increasing over I (LNS+)
- Hence, for any $\overline{U}$, there is a unique income M such that

$$\overline{U} = U(x(p, M))$$

- Inverting this, we can solve for $M(p, \overline{U})$

Dual Problem: Minimizing Expenditure

- In fact, minimizing expenditure yields:

$$\frac{p_1}{\frac{\partial U}{\partial x_1}} = \frac{p_2}{\frac{\partial U}{\partial x_2}} = \lambda$$

- Maximize Utility's FOC yields:

$$\frac{\frac{\partial U}{\partial x_1}}{p_1} = \frac{\frac{\partial U}{\partial x_2}}{p_2} = \lambda$$

- This close relationship between $x^c(p, \overline{U})$ and $x(p, I)$ indicates why they are "sisters"…

Substitution Effect for Compensated Demand

- Compensated Demand

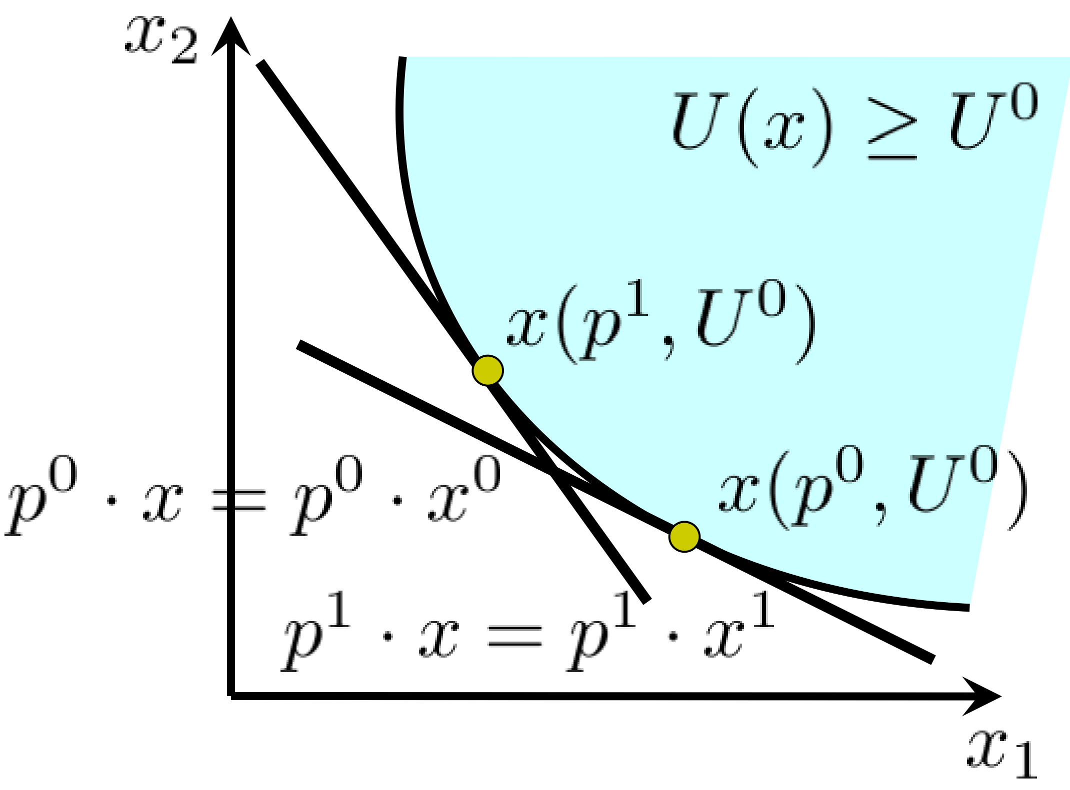
$$x^c(p,\overline{U}) \text{ solves } M(p,\overline{U}) = \min_x \{p \cdot x | U(x) \le \overline{U}\}$$

- By Envelope Theorem:
- Effect of Price Change $\dfrac{\partial M}{\partial p_j} = x_j^c(p, U^0)$
  - (Substitution Effect…)
  - How much more does Taiwan have to pay if the price of submarines increase (to maintain the same level of defense)?

Elasticity of Substitution (for Compensated Demand)

$$\sigma = \mathcal{E}\left(\frac{x_2^c}{x_1^c}, p_1\right)$$
$$= \mathcal{E}\left(\frac{x_1^c}{x_2^c}, p_2\right)$$

- The change in <u>consumption ratio</u> in response to a change in <u>prices</u>…



Why $\sigma = \mathcal{E}\left(\frac{x_2^c}{x_1^c}, p_1\right) = \mathcal{E}\left(\frac{x_1^c}{x_2^c}, p_2\right)$?

- On the indifference curve,

$$U\left(x_1^c(p,\overline{U}), x_2^c(p,\overline{U})\right) = \overline{U}$$

- Hence, $$\frac{\partial U}{\partial x_1}\frac{\partial x_1^c}{\partial p_i} + \frac{\partial U}{\partial x_2}\frac{\partial x_2^c}{\partial p_i} = 0$$

- By FOC, $$\frac{p_1}{\frac{\partial U}{\partial x_1}} = \frac{p_2}{\frac{\partial U}{\partial x_2}} \Rightarrow p_1\frac{\partial x_1^c}{\partial p_i} + p_2\frac{\partial x_2^c}{\partial p_i} = 0$$

- Since, $$\frac{\partial M}{\partial p_j} = x_j^c(p, U^0) \Rightarrow \frac{\partial x_1^c}{\partial p_2} = \frac{\partial^2 M}{\partial p_2 \partial p_1} = \frac{\partial x_2^c}{\partial p_1}$$

Why $\sigma = \mathcal{E}\left(\frac{x_2^c}{x_1^c}, p_1\right) = \mathcal{E}\left(\frac{x_1^c}{x_2^c}, p_2\right)$?

$$\sigma = \mathcal{E}\left(\frac{x_2^c}{x_1^c}, p_1\right) = \mathcal{E}\left(x_2^c, p_1\right) - \mathcal{E}\left(x_1^c, p_1\right)$$

$$= \frac{p_1}{x_2^c}\underline{\underline{\frac{\partial x_2^c}{\partial p_1}}} - \underline{\underline{\frac{p_1}{x_1^c}\frac{\partial x_1^c}{\partial p_1}}} = \underline{\underline{\frac{p_1}{x_2^c}\frac{\partial x_1^c}{\partial p_2}}} - \frac{1}{x_1^c}\cdot(-p_2)\underline{\underline{\frac{\partial x_2^c}{\partial p_1}}}$$

$$= -\frac{p_2}{x_2^c}\frac{\partial x_2^c}{\partial p_2} + \frac{p_2}{x_1^c}\frac{\partial x_1^c}{\partial p_2}$$

$$\boxed{\frac{\partial x_1^c}{\partial p_2} = \frac{\partial x_2^c}{\partial p_1}}$$

$$= -\mathcal{E}\left(x_2^c, p_2\right) + \mathcal{E}\left(x_1^c, p_2\right)$$

$$\boxed{p_1\frac{\partial x_1^c}{\partial p_i} + p_2\frac{\partial x_2^c}{\partial p_i} = 0}$$

$$= \mathcal{E}\left(\frac{x_1^c}{x_2^c}, p_2\right)$$

Elasticity of Substitution (for Compensated Demand)

- Proposition 2.2-3:

$$\sigma = \mathcal{E}\left(\frac{x_2^c}{x_1^c}, p_1\right) = \frac{\mathcal{E}(x_2^c, p_1)}{k_1}, \quad k_1 = \frac{p_1 x_1}{p \cdot x}$$

$$\left(\frac{\text{compensated cross price elasticity}}{\text{expenditure share}}\right)$$

$$= -\frac{\mathcal{E}(x_1^c, p_1)}{1 - k_1}$$

Elasticity of Substitution (for Compensated Demand)

- Proof of Proposition 2.2-3:

$$p_1 \frac{\partial x_1^c}{\partial p_i} + p_2 \frac{\partial x_2^c}{\partial p_i} = 0$$

$$k_1 \mathcal{E}(x_1^c, p_1) + k_2 \mathcal{E}(x_2^c, p_1)$$

$$= \frac{p_1 x_1}{I} \cdot \frac{p_1}{x_1} \frac{\partial x_1^c}{\partial p_1} + \frac{p_2 x_2}{I} \cdot \frac{p_1}{x_2} \frac{\partial x_2^c}{\partial p_1} = 0$$

$$\Rightarrow \sigma = \mathcal{E}(x_2^c, p_1) - \mathcal{E}(x_1^c, p_1)$$

$$= \mathcal{E}(x_2^c, p_1) \cdot \left(1 + \frac{k_2}{k_1}\right) = \frac{\mathcal{E}(x_2^c, p_1)}{k_1}$$

$$= \mathcal{E}(x_1^c, p_1) \cdot \left(-\frac{k_1}{k_2}\right) \cdot \frac{1}{k_1} = -\frac{\mathcal{E}(x_1^c, p_1)}{1 - k_1}$$

Elasticity of Substitution (for Compensated Demand)

- Verify that $\sigma = \theta$ for CES:
- Since $x_1 = \left(\frac{\alpha_1}{\alpha_2}\frac{p_2}{p_1}\right)^\theta \cdot x_2 \Rightarrow \frac{x_2}{x_1} = \left(\frac{\alpha_2}{\alpha_1} \cdot \frac{p_1}{p_2}\right)^\theta$

$$\Rightarrow \ln\left(\frac{x_2^c}{x_1^c}\right) = \theta(\ln p_1 - \ln p_2 + \ln \alpha_2 - \ln \alpha_1)$$

$$\Rightarrow \sigma = \mathcal{E}\left(\frac{x_2^c}{x_1^c}, p_1\right) = p_1 \cdot \frac{\partial}{\partial p_1}\left[\ln\left(\frac{x_2^c}{x_1^c}\right)\right]$$

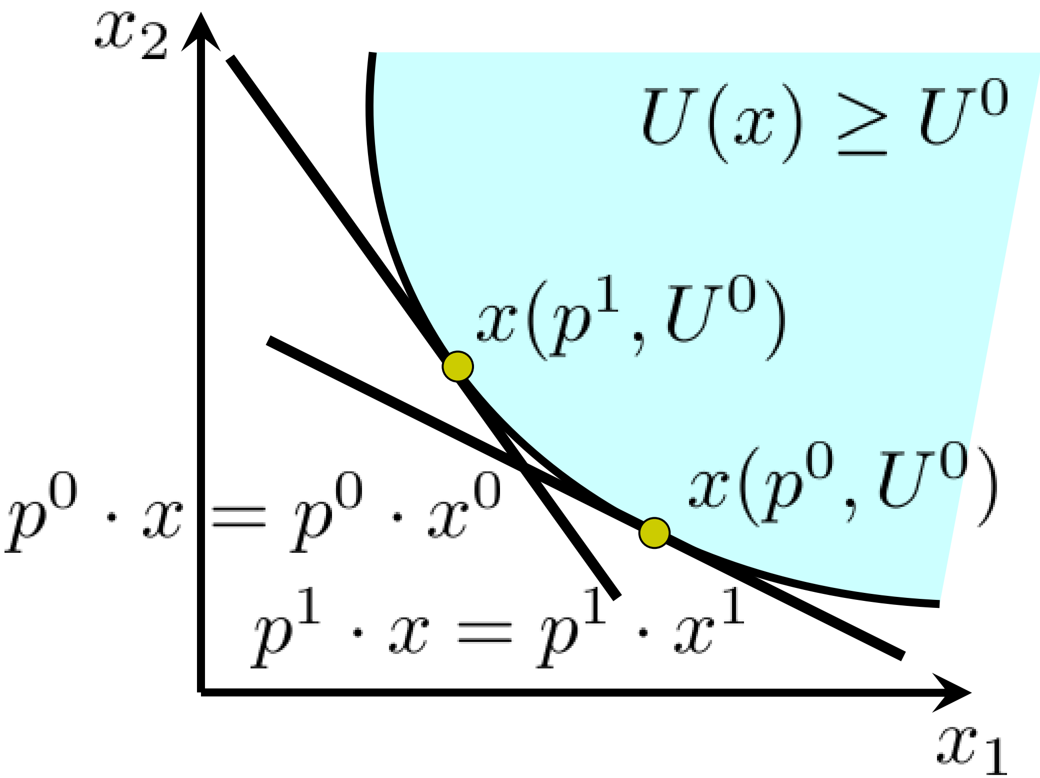
$$= p_1 \cdot \frac{\theta}{p_1} = \theta$$

Summary for Elasticity of Substitution

- 1. $\sigma = \mathcal{E}\left(\frac{x_2^c}{x_1^c}, p_1\right)$
- 2. $= \frac{\mathcal{E}(x_2^c, p_1)}{k_1}$

  $= -\frac{\mathcal{E}(x_1^c, p_1)}{1 - k_1}$

  $k_1 = \frac{p_1 x_1}{p \cdot x}$
- 3. $\sigma = \theta$ for CES...



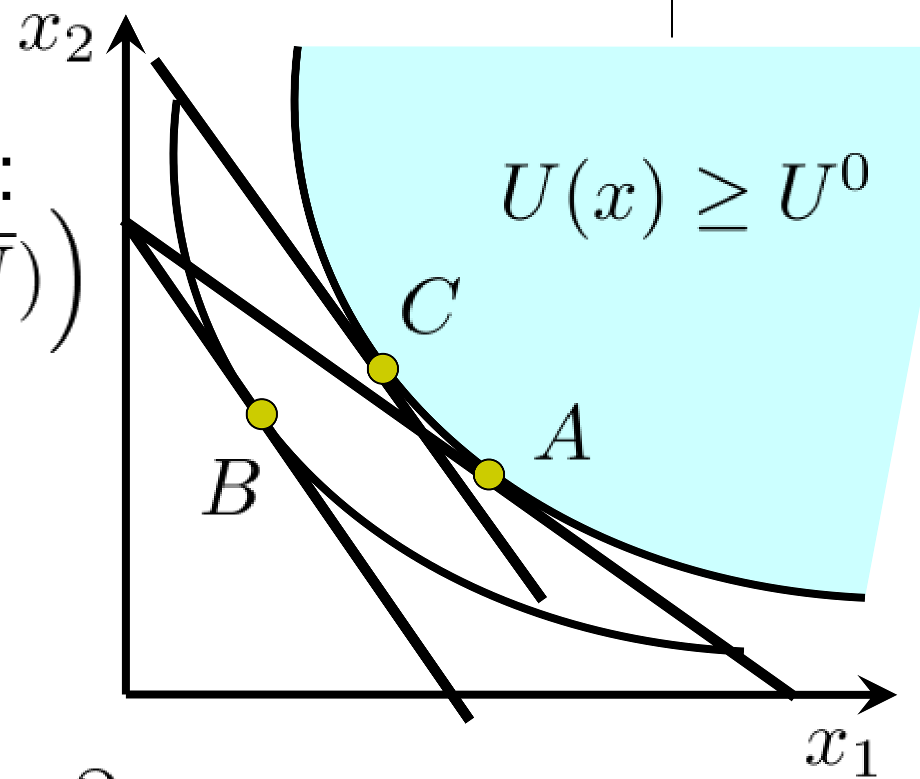
Total Price Effect = Income Effect + Substitution Effect

- For $M(p,\overline{U})$ & $x_1(p,I)$
- Compensated Demand:

$$x_1^c(p,\overline{U}) = x_1\left(p, M(p,\overline{U})\right)$$

$$\frac{\partial x_1^c}{\partial p_1} = \frac{\partial x_1}{\partial p_1} + \frac{\partial x_1}{\partial I} \cdot \frac{\partial M}{\partial p_1}$$

$$\left(\frac{\partial M}{\partial p_1} = x_1^c\right)$$



- Slutsky Equation:

$$\underbrace{\frac{\partial x_1}{\partial p_1}}_{A \to B} = \underbrace{\frac{\partial x_1^c}{\partial p_1}}_{A \to C} \underbrace{- x_1 \cdot \frac{\partial x_1}{\partial I}}_{C \to B}$$

Total Price Effect = Income Effect + Substitution Effect

- Slutsky Equation:

$$\frac{\partial x_1}{\partial p_1} = \frac{\partial x_1^c}{\partial p_1} - x_1 \cdot \frac{\partial x_1}{\partial I}$$

- Elasticity Version:

$$\frac{p_1}{x_1}\frac{\partial x_1}{\partial p_1} = \frac{p_1}{x_1}\frac{\partial x_1^c}{\partial p_1} - \frac{p_1 x_1}{I}\frac{I}{x_1} \cdot \frac{\partial x_1}{\partial I}$$

- Or,

$$\mathcal{E}(x_1, p_1) = \mathcal{E}(x_1^c, p_1) - k_1 \cdot \mathcal{E}(x_1, I)$$

Summary of 2.2

- Consumer Problem: Maximize Utility
- Income Effect
- Dual Problem: Minimize Expenditure
- Substitution Effect:
  - =Compensated Price Effect
  - Elasticity of Substitution
- Total Price Effect:
  - = Compensated Price Effect + Income Effect

Summary of 2.2

- Homework:
- Riley 2.2-4, 5, 6
- J/R – 1.17, 1.18, 1.27, 1.37, 1.43, 1.50, 1.53