Envelope Theorem

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(Lecture 2, Micro Theory I)

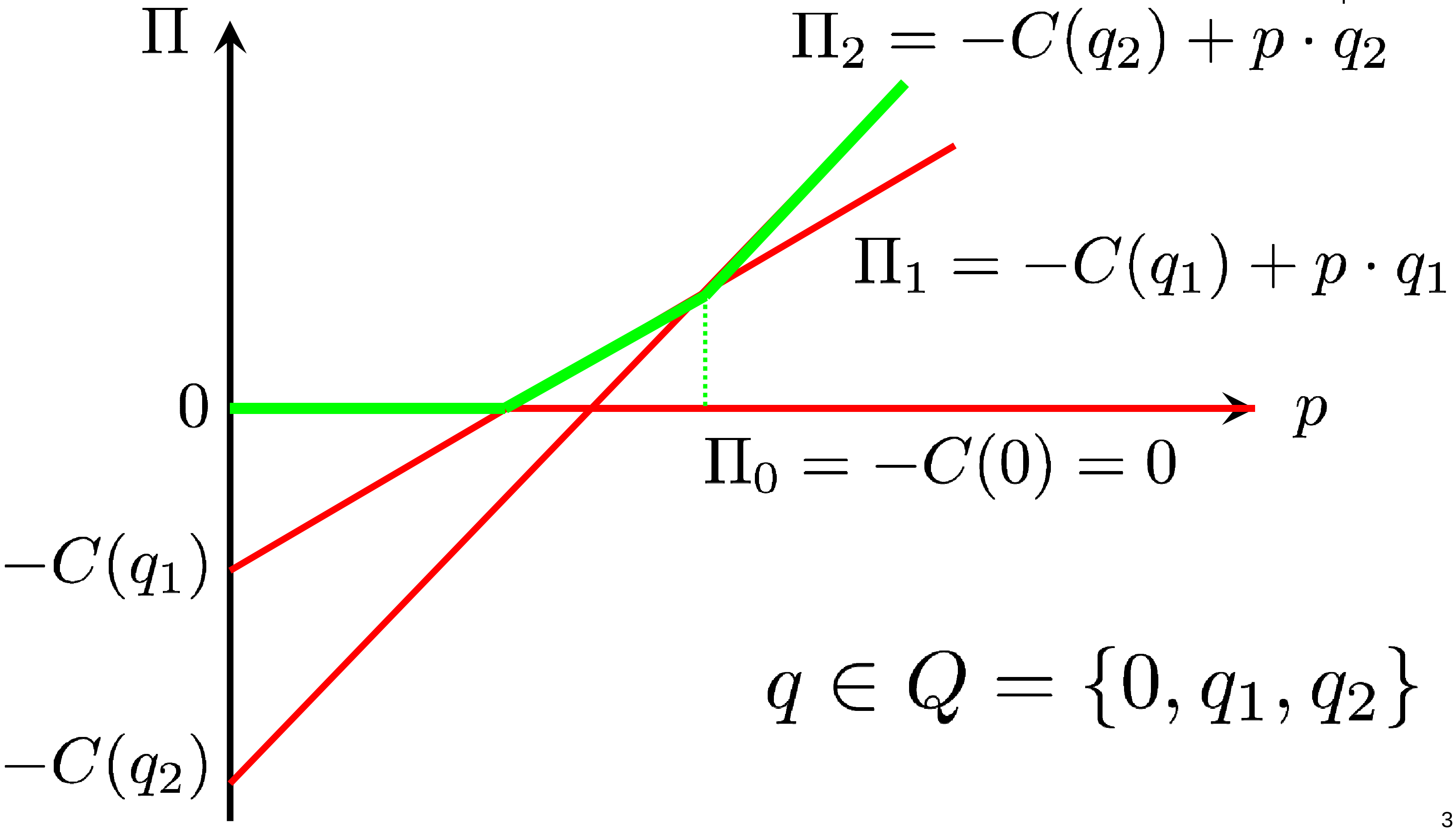
When Environment Changes, How Do Firms Adjust?

- A price-taking firm has cost $C(q)$
  - Can sell as much as it wishes at fix price p
- Profit is $\pi = p \cdot q - C(q)$
- Given a change in prices p, how would profit change (as the firm re-optimizes output q)?
  - Direct Effect: $\Delta p \cdot q$
  - Indirect Effect: $\Delta\pi \text{ due to } q \to q'$
- First assume only three possible outputs…

$$q \in Q = \{0, q_1, q_2\}$$

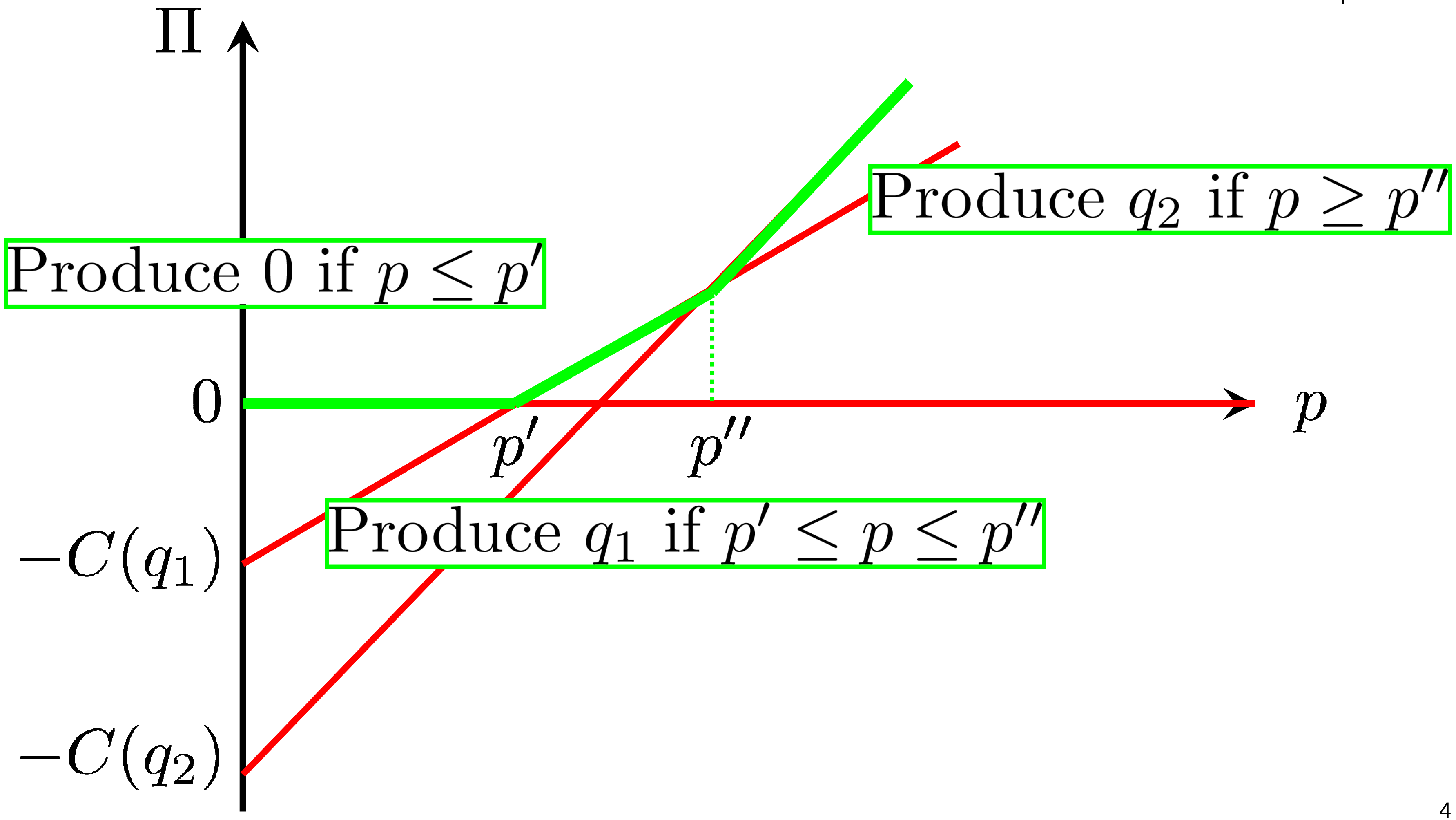
  - Profit is straight line for each possible output

Three Output States

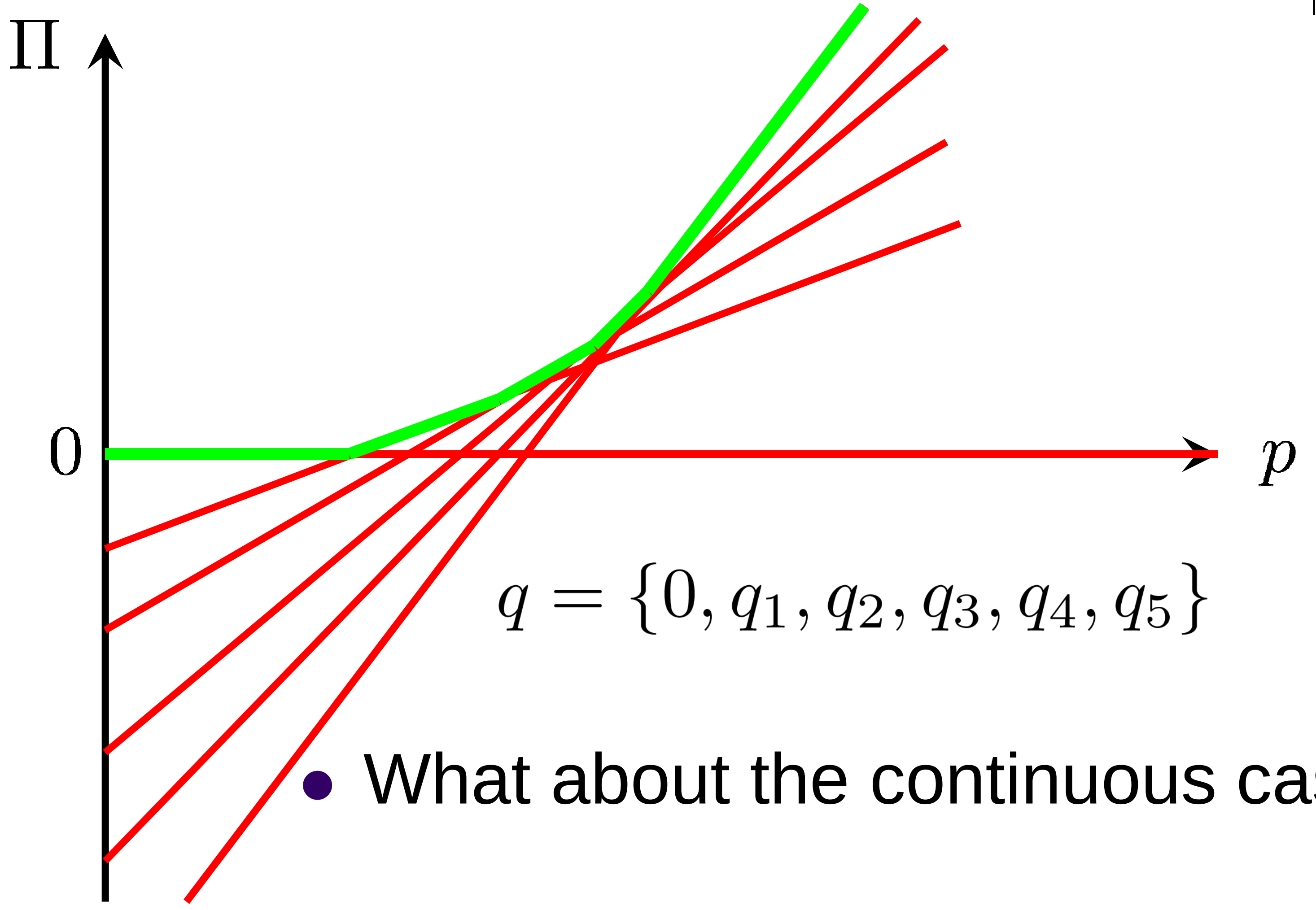


$$q \in Q = \{0, q_1, q_2\}$$

Upper Envelope for Three Output States

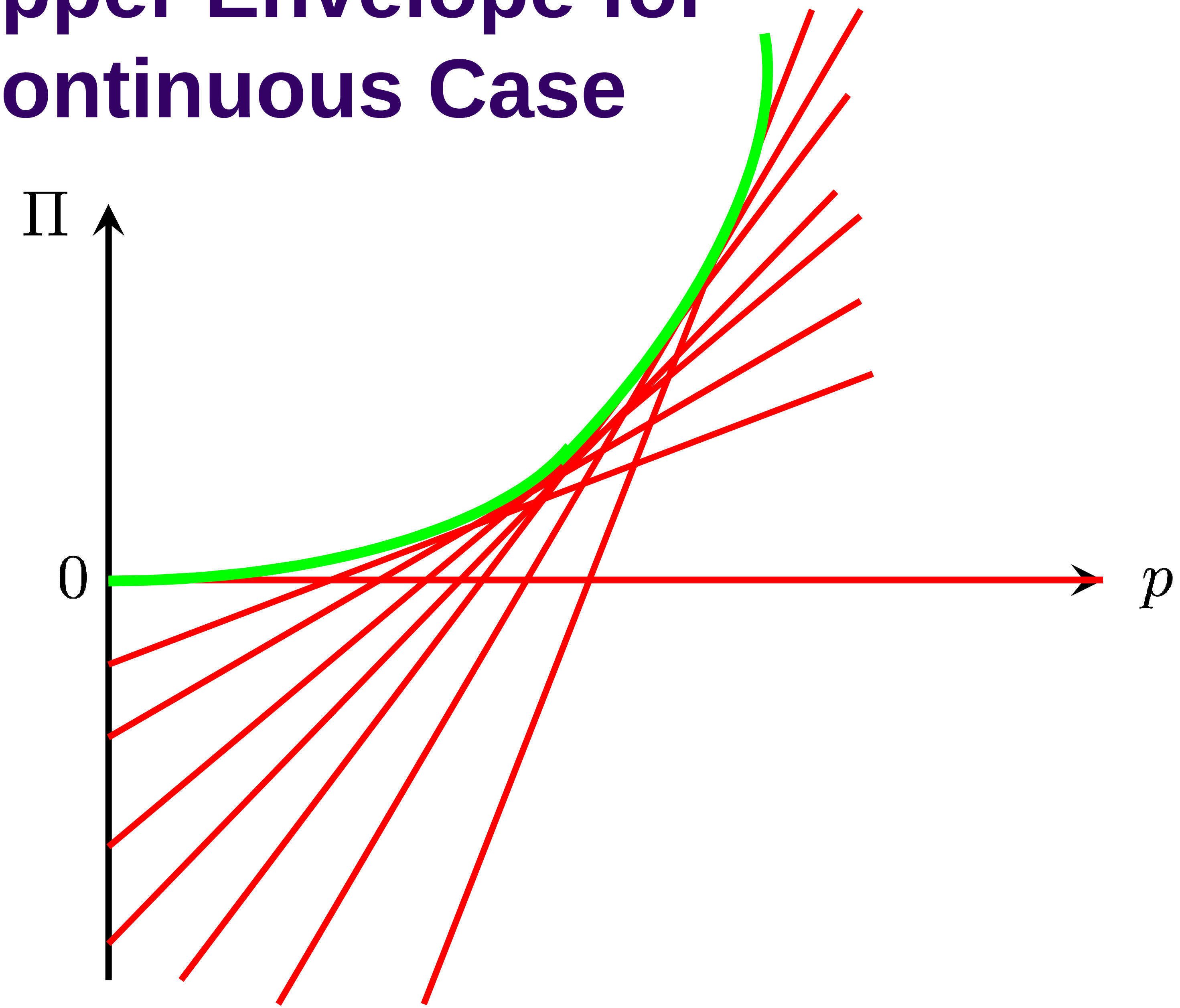


Upper Envelope for Six Output States



- What about the continuous case?

Upper Envelope for Continuous Case



When Environment Changes, How Do Firms Adjust?

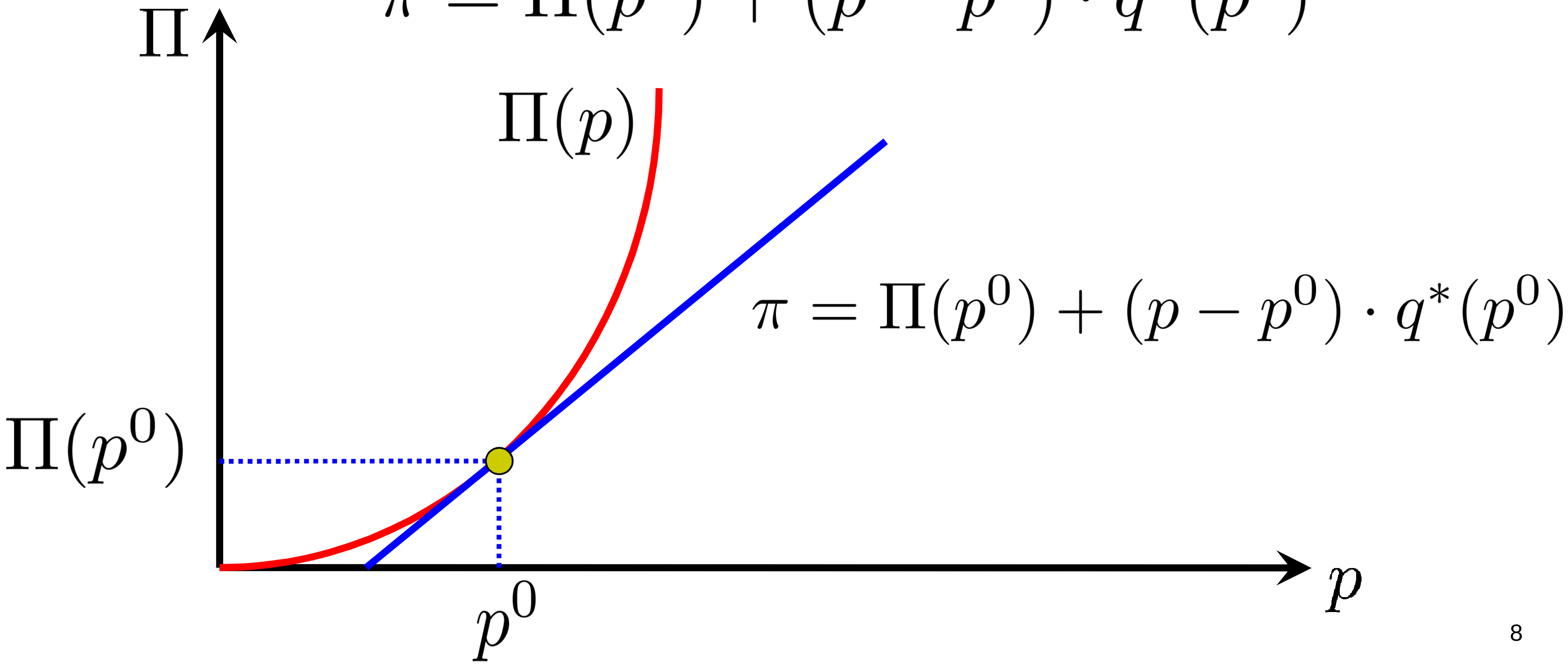
- Output can be any real number
- Firm solves $q^*(p)$ to $\max\{\pi = p \cdot q - C(q)\}$
- Maximized profit is $\Pi(p) = p \cdot q^*(p) - C\left(q^*(p)\right)$
- Initial output price p^0 (fixed)
  - Initial output $q^*(p^0)$
  - Initial profit $\Pi(p^0)$
- Profit (with fixed output) is

$$\pi = \Pi(p^0) + (p - p^0) \cdot q^*(p^0)$$

When Environment Changes, How Do Firms Adjust?

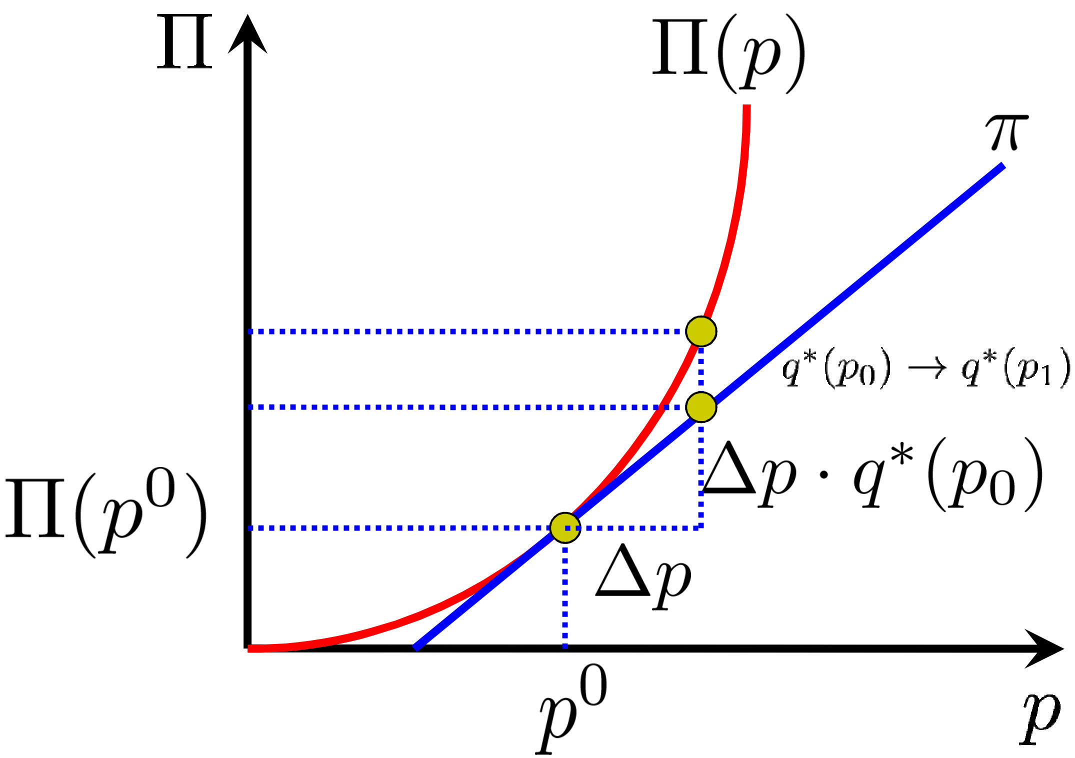
- Fixing output, increase in price changes profit by $q^*(p)$ per dollar, so (fixed output) profit is

$$\pi = \Pi(p^0) + (p - p^0) \cdot q^*(p^0)$$



When Environment Changes, How Do Firms Adjust?

$$\frac{\partial \Pi}{\partial p}(p^0) = q^*(p^0)$$

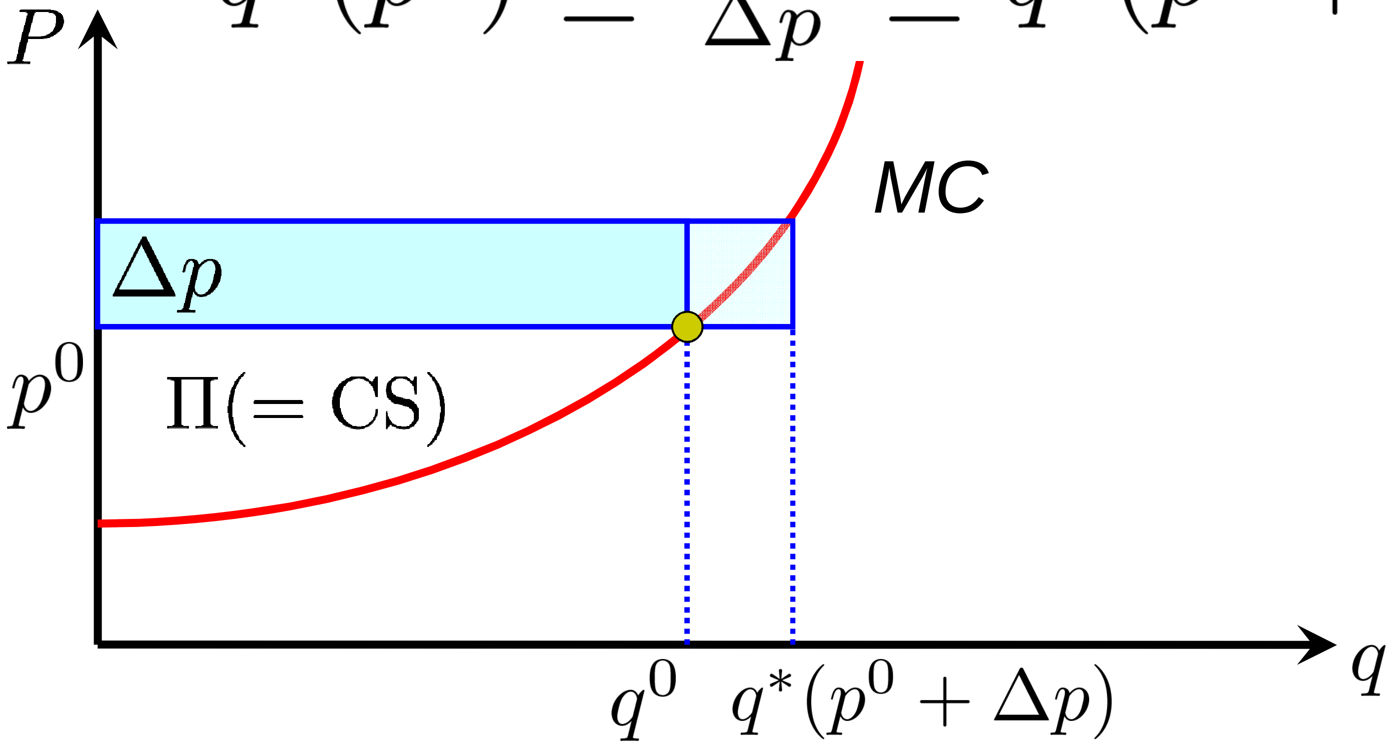


- Firm cannot be worse off if it can change quantity
- $\Pi(p)$ is above π
  - Tangent to π if $\Pi(p)$ smooth
- Total effect = Direct effect only
  - Ignore indirect eff.

Another Graphic Presentation
(P-q instead of π-p)

$$q^*(p^0)\Delta p \leq \Delta\Pi \leq q^*(p^0 + \Delta p)\Delta p$$

$$q^*(p^0) \leq \frac{\Delta\Pi}{\Delta p} \leq q^*(p^0 + \Delta p)$$



In fact, we have Prop. 1.3-3: Envelope Theorem I

- Assume:
  - X is closed and bounded, (Feasible output)
  - f is continuously differentiable (Profit)

  $x(\alpha) = \arg\max_{x \in X} \{f(x, \alpha)\}$ is unique, $q^*(p^0)$

  $\alpha = p$
- Then, value function, $F(\alpha) = \max_{x \in X} \{f(x, \alpha)\}$

  $\Pi(p)$

  is differentiable and $\frac{dF}{d\alpha} = \frac{\partial f}{\partial \alpha}(x(\alpha), \alpha)$.

  (Only Direct Effect)

Proposition 1.3-3: Envelope Theorem I

- Direct Effect = Total Effect (at the margin)
- This only allows the maximand to be affected by the parameter change…
- To allow for both the maximand and the constraints to be affected by the parameter change, need slightly stronger assumptions…

Proposition 1.3.4: Envelope Theorem II

- For $F(\alpha) = \max_x \{f(x,\alpha) | h(x,\alpha) \geq 0, x \geq 0\}$

$$\mathbf{L}(x,\lambda,\alpha) = f(x,\alpha) + \lambda h(x,\alpha)$$

- Suppose:
  - f and h are continuously differentiable
  - $x(\alpha), \lambda(\alpha)$ unique solutions; CQ hold.
  - $x(\alpha)$ and $\lambda(\alpha)$ continuously differentiable
  - at α^0 (implicit function theorem applies)
- Then, $\dfrac{\partial F}{\partial \alpha}(\alpha^0) = \dfrac{\partial \mathbf{L}}{\partial \alpha}\left(x(\alpha^0), \lambda(\alpha^0), \alpha\right).$

Example: Hunghai (not to be mistaken as Foxconn Tech. Group…)

- Hunghai is a price-taking firm making jpods
  - Sell 3,000 jpods to Pineapple at price $p_i = \$100$
  - Total Cost is $C(q) = \$180,000$
- What is the elasticity of profit w.r.t. price
  - If output is held fixed?
  - If Hung-Hai responds optimally to price change?

$$\epsilon(\Pi, p_i) = \frac{p_i}{\Pi} \frac{\partial \Pi}{\partial p_i}$$

- Hunghai sees a new opportunity and sells 1,500 Vii's to Rentientang at price $p_w = \$200$
  - Production of jpods drop to 2,400, total cost rises to \$300,000. Can you calculate the new $\epsilon(\Pi, p_i)$?

Example: Hunghai

- Hunghai is a price-taking firm making jpods
  - Sell 3,000 jpods to Pineapple at price $p_i = \$100$
  - Total Cost is $C(q) = \$180,000$

$$\begin{aligned} \Pi &= p \times q - C(q) \\ &= \$100 \times 3,000 - \$180,000 = \$120,000 \end{aligned}$$

$$\frac{\partial \Pi}{\partial p_i} = q_i = 3,000 \Rightarrow \epsilon = \frac{p_i}{\Pi}\frac{\partial \Pi}{\partial p_i} = \frac{\$100 \cdot 3,000}{\$120,000} = \frac{5}{2}$$

- Hunghai's elasticity of profit wrt. jpod price is 2.5 for both fixed and variable output (by ET!)

Example: Hunghai

- Hunghai sees a new opportunity and sells 1,500 Vii's to Rentientang at price $p_w = \$200$
  - Production of jpods drop to 2,400, price $p_i = \$100$
  - total cost rises to \$300,000. Calculate new $\epsilon(\Pi, p_i)$?

$$\begin{aligned}\Pi &= \$100 \times 2,400 + \$200 \times 1,500 - \$300,000 \\ &= \$240,000\end{aligned}$$

$$\frac{\partial \Pi}{\partial p_i} = q_i' = 2,400 \Rightarrow \epsilon = \frac{p_i}{\Pi}\frac{\partial \Pi}{\partial p_i} = \frac{\$100 \cdot 2,400}{\$240,000} = 1$$

What does this all mean?

- Hunghai used to only produce jpods
- Since it is a price-taker, if Pineapple Corp. decides to lower prices by 10%, Hunghai's profit would decrease by 25%
- Even if Hunghai tries to re-optimize! (ET)
- After diversifying to producing also Vii's, it's profit is now less prone to Pineapple's price cuts (lowers by 10% if prices are cut by 10%)
- Isn't this what firms in Hsinchu Science Park do?

Summary of 1.3

- Re-maximize under environmental change
  - Direct Effect: Change in profit (objective function)
  - Indirect Effect: Change due to re-optimization
- Envelope Theorem(s):
  - Only have Direct Effect at the margin

- Homework: Exercise 1.3-1, 3, 4