Equilibrium Futures Prices

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(Lecture 14, Micro Theory I)

What We Learned about Equilibrium?

- Pareto Efficient Allocation (PEA) – Optimum
  - Cannot make one better off without hurting others
- Walrasian Equilibrium (WE) – Price Vector
  - When Supply Meets Demand
  - Even if there is only one person Robinson Crusoe
- 1st Welfare Theorem: WE are PEA
  - Schizophrenic Robinson Crusoe achieves optimum
- 2nd Welfare Theorem: PEA supported as WE
- Also works under inter-temporal choices…

Walrasian Equilibrium

- Price-taking: $\text{Prices } p > 0$
- Firm f chooses production plan $y^f(p)$ so it solves
$$\max_y \left\{ p \cdot y^f \middle| y^f \in \gamma^f \right\}, f = 1, \cdots, F$$
- Consumer h has θ^{hf} ownership shares in firm f and earns dividends equal to $\Pi^f(p) = p \cdot y^f(p)$
- Consumer h chooses consumption $x^h(p)$ so it solves
$$\max_x \left\{ U^h(x^h) \middle| p \cdot x^h \le p \cdot \omega^h + \sum_{f=1}^{F} \theta^{hf} \Pi^f \right\}$$

Reinterpreting the General Model as Spot & Futures

- 2 Periods: *t=1, 2*
- Firm's Production Plans $y^f = (y_1^f, y_2^f)$
  - Each period: $y_t^f = (y_{t1}^f, \cdots, y_{tn}^f), t = 1, 2$
- Consumer's Consumption Vectors: $x^h = (x_1^h, x_2^h)$
- Price vector: $p = (p_1, p_2)$
- All trade is done in Period 1:
- Spot price vector: $p_1 = (p_{11}, \cdots, p_{1n})$
- Futures price vector: $p_2 = (p_{21}, \cdots, p_{2n})$

Reinterpreting Spot & Futures as Borrowing & Lending

- Market interest rate: r ; Period 2 Spot price: p_2^s
- Firm's Dividends and Borrowing:
- Period 1: $d_1^f = p_1 \cdot y_1^f + B_1^f$
- Period 2: $d_2^f = p_2^s \cdot y_2^f - B_1^f(1+r)$
- PV is $d_1^f + \frac{d_2^f}{1+r} = p_1 \cdot y_1^f + \frac{1}{1+r} p_2^s \cdot y_2^f$
- Relationship with Futures: $p_2^s = (1+r)p_2$

$$p_1 \cdot y_1^f + p_2 \cdot y_2^f = \Pi^f(p)$$

Reinterpreting Spot & Futures as Borrowing & Lending

- Consumer's Budget Constraint and Saving:
- Period 1: $p_1 \cdot x_1^h \leq p_1 \cdot \omega_1^h + \sum_{f=1}^{F} \theta^{hf} d_1^f - S_1^h$
- Period 2: $p_2^s \cdot x_2^h \leq p_2^s \cdot \omega_2^h + \sum_{f=1}^{F} \theta^{hf} d_2^f + (1+r) S_1^h$
- PV is $p_1 \cdot x_1^h + \frac{1}{1+r} p_2^s \cdot x_2^h$

$$\leq p_1 \cdot \omega_1^h + \frac{1}{1+r} p_2^s \cdot \omega_2^h + \sum_{f=1}^{F} \theta^{hf} \left[\quad \Pi^f(p) \quad \right]$$

Rational Expectations

- For this to work, we need:

1. Rational Expectations: Agents correctly forecast equilibrium future spot prices
   - May not be true, but okay if arbitrageurs fix it…
2. No Bankruptcy: Consumers have to live up to their promises for lenders to lend to them
3. No Uncertainty (about future preferences and technology): Can be extended in Ch. 7.

Example

- Two goods, two periods: $x_t = (x_{t1}, x_{t2})$
- All consumers have same log preferences
- $U = u(x_1) + \frac{1}{2}u(x_2)$ where $u(x_t) = 2\ln x_{t1} + \ln x_{t2}$
- Endowments: $\omega_1 = (120, 120), \quad \omega_2 = (0, 0)$
- Technology: Investment z_{1i} to earn q_{2i} later…

$$q_{21} = 2z_{11}, \quad q_{22} = 4z_{12}$$

- Solve for optimal consumption, Walrasian Equilibrium spot and future prices (and p_2^s)

Example: Optimal Consumption

$$U = [2\ln x_{11} + \ln x_{12}] + \frac{1}{2}[2\ln x_{21} + \ln x_{22}]$$

- Can use Representative Agent
  - Since preferences are homothetic and identical
- Note $x_{11} = \omega_{11} - z_{11},\ x_{12} = \omega_{12} - z_{12},$
  $x_{21} = 2z_{11}, \quad x_{22} = 4z_{12}$
- RA solves

$$U = 2\ln(120 - z_{11}) + \ln(120 - z_{12})$$
$$+ \ln(2z_{11}) + \frac{1}{2}\ln(4z_{12})$$

Example: Optimal Consumption

$$U = 2\ln(120 - z_{11}) + \ln(120 - z_{12}) + \ln(2z_{11}) + \frac{1}{2}\ln(4z_{12})$$

- FOC: (interior)

$$\frac{\partial U}{\partial z_{11}} = \frac{-2}{120 - z_{11}} + \frac{2}{2z_{11}} = 0$$

$$\frac{\partial U}{\partial z_{12}} = \frac{-1}{120 - z_{12}} + \frac{4}{8z_{12}} = 0$$

- Hence,
- $(z_{11}^*, z_{12}^*) = (40, 40)$; $(x_{11}^*, x_{12}^*) = (80, 80)$ $(x_{21}^*, x_{22}^*) = (80, 160)$

Example: Walrasian Equilibrium

- 2^{nd} Welfare Theorem applies
  - Since technology and preferred sets convex
- Prices for delivery in period 1: $p_1 = (p_{11}, p_{12})$
  - Futures prices for delivery in period 2: $p_2 = (p_{21}, p_{22})$
- Given consumer wealth *W*, choose *x* to solve

$$\max_x \left\{ U(x) \middle| p \cdot x = p_1 \cdot x_1 + p_2 \cdot x_2 = W \right\}$$

- FOC: $$\frac{\frac{\partial U}{\partial x_{11}}}{p_{11}} = \frac{\frac{\partial U}{\partial x_{12}}}{p_{12}} = \frac{\frac{\partial U}{\partial x_{21}}}{p_{21}} = \frac{\frac{\partial U}{\partial x_{22}}}{p_{22}} = \lambda$$

Example: Walrasian Equilibrium

$$U = [2\ln x_{11} + \ln x_{12}] + \frac{1}{2}[2\ln x_{21} + \ln x_{22}]$$

- Thus, at $(x^*_{11}, x^*_{12}) = (80, 80)$ $(x^*_{21}, x^*_{22}) = (80, 160)$

$$\frac{2}{p_{11}80} = \frac{1}{p_{12}80} = \frac{1}{p_{21}80} = \frac{\frac{1}{2}}{p_{22}160} = \lambda$$

- Set numeraire $p_{11} = 1$,
- Spot prices $p_1 = (p_{11}, p_{12}) = (1, \frac{1}{2})$
- Future prices $p_2 = (p_{21}, p_{22}) = (\frac{1}{2}, \frac{1}{8})$

Example: Walrasian Equilibrium

- Set numeraire for all periods: $p_{11} = p_{21}^s = 1$
- Since spot prices are $p_1 = (p_{11}, p_{12}) = (1, \frac{1}{2})$
- future prices are $p_2 = (p_{21}, p_{22}) = (\frac{1}{2}, \frac{1}{8})$
- In period 2, future spot prices $p_2^s = (1, \frac{1}{4})$
- With nominal interest rate r
- Arbitrageurs equate $p_{21}(1 + r) = p_{21}^s$
- Hence, $r = 1$

Summary of 5.5

- Time can be incorporated as a Day 1 Market
  - Spot and Future prices
- Or, as a Rational Expectation Equilibrium with Borrowing and Lending
  - may not hold if:
  1. No arbitrageurs to ensure expectations are rational
  2. Consumers could go bankrupt to avoid repaying
  3. Uncertain about future preferences or technology
- Homework: Riley - 5.5-1~5; 2008 finals Q3