

# Equilibrium Futures Prices

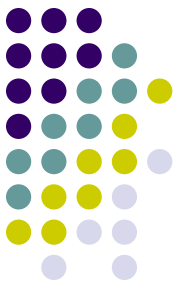
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2009/12/11

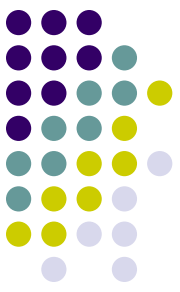
(Lecture 14, Micro Theory I)



# What We Learned about Equilibrium?



- Pareto Efficient Allocation (PEA) – Optimum
  - Cannot make one better off without hurting others
- Walrasian Equilibrium (WE) – Price Vector
  - When Supply Meets Demand
  - Even if there is only one person Robinson Crusoe
- 1<sup>st</sup> Welfare Theorem: WE are PEA
  - Schizophrenic Robinson Crusoe achieves optimum
- 2<sup>nd</sup> Welfare Theorem: PEA supported as WE
- Also works under inter-temporal choices...

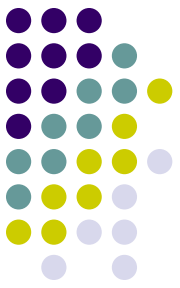


# Walrasian Equilibrium

- **Price-taking:** Prices  $p > 0$
- **Firm  $f$**  chooses production plan  $y^f(p)$  so it solves
$$\max_y \{ p \cdot y^f \mid y^f \in \gamma^f \}, f = 1, \dots, F$$
- **Consumer  $h$**  has  $\theta^{hf}$  ownership shares in firm  $f$  and earns dividends equal to  $\Pi^f(p) = p \cdot y^f(p)$
- **Consumer  $h$**  chooses consumption  $x^h(p)$  so it solves

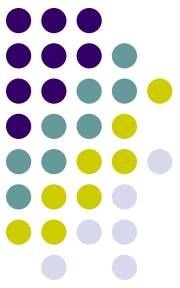
$$\max_x \left\{ U^h(x^h) \mid p \cdot x^h \leq p \cdot \omega^h + \sum_{f=1}^F \theta^{hf} \Pi^f \right\}$$

# Reinterpreting the General Model as Spot & Futures



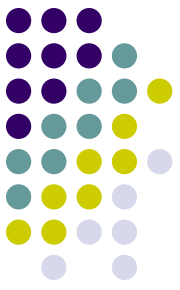
- 2 Periods:  $t=1, 2$
- Firm's Production Plans  $y^f = (y_1^f, y_2^f)$ 
  - Each period:  $y_t^f = (y_{t1}^f, \dots, y_{tn}^f), t = 1, 2$
- Consumer's Consumption Vectors:  $x^h = (x_1^h, x_2^h)$
- Price vector:  $p = (p_1, p_2)$
- All trade is done in Period 1:
- Spot price vector:  $p_1 = (p_{11}, \dots, p_{1n})$
- Futures price vector:  $p_2 = (p_{21}, \dots, p_{2n})$

# Reinterpreting Spot & Futures as Borrowing & Lending



- Market interest rate:  $r$  ; Period 2 Spot price:  $p_2^s$
- Firm's Dividends and Borrowing:
- Period 1:  $d_1^f = p_1 \cdot y_1^f + B_1^f$
- Period 2:  $d_2^f = p_2^s \cdot y_2^f - B_1^f(1 + r)$
- PV is  $d_1^f + \frac{d_2^f}{1 + r} = p_1 \cdot y_1^f + \frac{1}{1 + r} p_2^s \cdot y_2^f$
- Relationship with Futures:  $p_2^s = (1 + r)p_2$   
 $p_1 \cdot y_1^f + p_2 \cdot y_2^f = \Pi^f(p)$

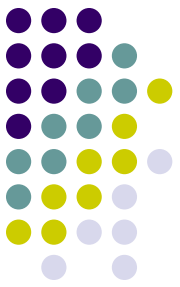
# Reinterpreting Spot & Futures as Borrowing & Lending



- **Consumer's Budget Constraint and Saving:**

- Period 1: 
$$p_1 \cdot x_1^h \leq p_1 \cdot \omega_1^h + \sum_{f=1}^F \theta^{hf} d_1^f - S_1^h$$
- Period 2: 
$$p_2^s \cdot x_2^h \leq p_2^s \cdot \omega_2^h + \sum_{f=1}^F \theta^{hf} d_2^f + (1+r)S_1^h$$
- PV is 
$$p_1 \cdot x_1^h + \frac{1}{1+r} p_2^s \cdot x_2^h$$

$$\leq p_1 \cdot \omega_1^h + \frac{1}{1+r} p_2^s \cdot \omega_2^h + \sum_{f=1}^F \theta^{hf} \left[ \Pi^f(p) \right]_6$$



# Rational Expectations

- For this to work, we need:
  1. **Rational Expectations:** Agents correctly forecast equilibrium future spot prices
    - May not be true, but okay if arbitrageurs fix it...
  2. **No Bankruptcy:** Consumers have to live up to their promises for lenders to lend to them
  3. **No Uncertainty** (about future preferences and technology): Can be extended in Ch. 7.



# Example

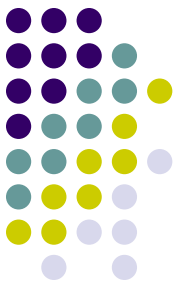
- Two goods, two periods:  $x_t = (x_{t1}, x_{t2})$
- All consumers have same log preferences
- $U = u(x_1) + \frac{1}{2}u(x_2)$  where  $u(x_t) = 2 \ln x_{t1} + \ln x_{t2}$
- Endowments:  $\omega_1 = (120, 120)$ ,  $\omega_2 = (0, 0)$
- Technology: Investment  $z_{1i}$  to earn  $q_{2i}$  later...

$$q_{21} = 2z_{11}, \quad q_{22} = 4z_{12}$$

- Solve for optimal consumption, Walrasian Equilibrium spot and future prices (and  $p_2^s$  )



# Example: Optimal Consumption



$$U = [2 \ln x_{11} + \ln x_{12}] + \frac{1}{2} [2 \ln x_{21} + \ln x_{22}]$$

- Can use Representative Agent
  - Since preferences are homothetic and identical

- Note  $x_{11} = \omega_{11} - z_{11}$ ,  $x_{12} = \omega_{12} - z_{12}$ ,  
 $x_{21} = 2z_{11}$ ,  $x_{22} = 4z_{12}$

- RA solves

$$U = 2 \ln(120 - z_{11}) + \ln(120 - z_{12}) \\ + \ln(2z_{11}) + \frac{1}{2} \ln(4z_{12})$$

# Example: Optimal Consumption



$$U = 2 \ln(120 - z_{11}) + \ln(120 - z_{12}) \\ + \ln(2z_{11}) + \frac{1}{2} \ln(4z_{12})$$

- FOC: (interior)

$$\frac{\partial U}{\partial z_{11}} = \frac{-2}{120 - z_{11}} + \frac{2}{2z_{11}} = 0$$

$$\frac{\partial U}{\partial z_{12}} = \frac{-1}{120 - z_{12}} + \frac{4}{8z_{12}} = 0$$

- Hence,

- $(z_{11}^*, z_{12}^*) = (40, 40)$ ;  $(x_{11}^*, x_{12}^*) = (80, 80)$   
 $(x_{21}^*, x_{22}^*) = (80, 160)$

# Example:

## Walrasian Equilibrium



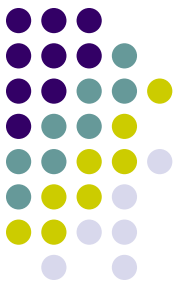
- 2<sup>nd</sup> Welfare Theorem applies
  - Since technology and preferred sets convex
- Prices for delivery in period 1:  $p_1 = (p_{11}, p_{12})$ 
  - Futures prices for delivery in period 2:  $p_2 = (p_{21}, p_{22})$
- Given consumer wealth  $W$ , choose  $x$  to solve

$$\max_x \{ U(x) \mid p \cdot x = p_1 \cdot x_1 + p_2 \cdot x_2 = W \}$$

- FOC: 
$$\frac{\frac{\partial U}{\partial x_{11}}}{p_{11}} = \frac{\frac{\partial U}{\partial x_{12}}}{p_{12}} = \frac{\frac{\partial U}{\partial x_{21}}}{p_{21}} = \frac{\frac{\partial U}{\partial x_{22}}}{p_{22}} = \lambda$$

# Example:

## Walrasian Equilibrium



$$U = [2 \ln x_{11} + \ln x_{12}] + \frac{1}{2} [2 \ln x_{21} + \ln x_{22}]$$

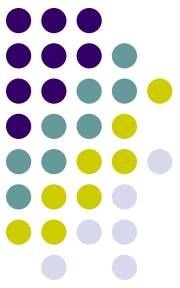
- Thus, at  $(x_{11}^*, x_{12}^*) = (80, 80)$   $(x_{21}^*, x_{22}^*) = (80, 160)$

$$\frac{2}{p_{11}80} = \frac{1}{p_{12}80} = \frac{1}{p_{21}80} = \frac{\frac{1}{2}}{p_{22}160} = \lambda$$

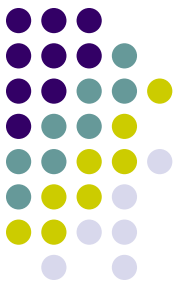
- Set numeraire  $p_{11} = 1$ ,
- Spot prices  $p_1 = (p_{11}, p_{12}) = (1, \frac{1}{2})$
- Future prices  $p_2 = (p_{21}, p_{22}) = (\frac{1}{2}, \frac{1}{8})$

# Example:

## Walrasian Equilibrium



- Set numeraire for all periods:  $p_{11} = p_{21}^s = 1$
- Since spot prices are  $p_1 = (p_{11}, p_{12}) = (1, \frac{1}{2})$
- future prices are  $p_2 = (p_{21}, p_{22}) = (\frac{1}{2}, \frac{1}{8})$
- In period 2, future spot prices  $p_2^s = (1, \frac{1}{4})$
- With nominal interest rate  $r$
- Arbitrageurs equate  $p_{21}(1 + r) = p_{21}^s$
- Hence,  $r = 1$



# Summary of 5.5

- **Time** can be incorporated as a Day 1 Market
  - Spot and Future prices
- Or, as a Rational Expectation Equilibrium with Borrowing and Lending
  - may not hold if:
    1. No arbitrageurs to ensure expectations are rational
    2. Consumers could go bankrupt to avoid repaying
    3. Uncertain about future preferences or technology
- Homework: Riley - 5.5-1~5; 2008 finals Q3