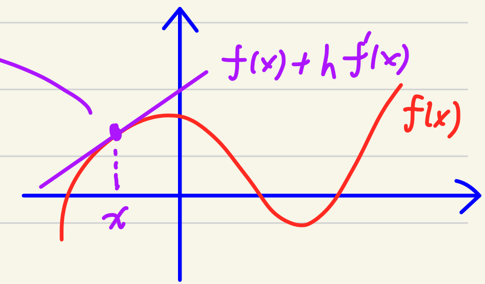
Online Math Camp (23S)

TA Session Note ($5/29$)

Taylor's Theorem:

A way to approximate functions with polynomials

$$\begin{cases} f(x+h) \approx f(x) + h f'(x) \\ f(x+h) = f(x) + h f'(c) \quad \text{by MVT} \end{cases}$$



$\Rightarrow$ How can we make this approximation more accurate?

$$f(x+h) \approx f(x) + A_1(h) f'(x) + A_2(h) f''(x) + \cdots + A_n(h) f^{(n)}(x) \quad \text{(Goal)}$$

Ex: $f(x) = x^2$ at $x=0$: $f(0+h) = h^2 \approx f(0) + A_1(h) f'(0) + A_2(h) f''(0) + \cdots + A_n(h) f^{(n)}(0)$

$$= 0 + 0 + 2A_2(h) + 0 + \cdots + 0$$

$$\Rightarrow A_2(h) = \frac{h^2}{2}$$

Similarly, with $f(x) = x^n$ at $x=0$, we obtain $A_n(h) = \frac{1}{n!} h^n$.

Thm If $f^{(n-1)}$ is continuous on $[a,b]$ and $f^{(n)}$ exists on (a,b).

Then, $f(x+h) \approx \underline{f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(x)}$

$$= \underline{f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(x)} + \frac{h^n}{n!}f^{(n)}(c),\ c \in (a,b)$$

i.e. $f(x) = f(a) + \sum_{i=1}^{n-1} \frac{1}{i!}(x-a)^i f^{(i)}(a) + \frac{1}{n!}(x-a)^n f^{(n)}(\tilde{a}),\ \tilde{a} \in (0,a)$

(pf) $\begin{cases} F(x) = f(x) - \left[f(a) + \sum_{i=1}^{n-1} \frac{1}{i!}(x-a)^i f^{(i)}(a)\right] \\ G(x) = (x-a)^n \end{cases} \Rightarrow \begin{cases} F(a) = F'(a) = \cdots = F^{(n-1)}(a) = 0 \\ G(a) = G'(a) = \cdots = G^{(n-1)}(a) = 0 \end{cases}$

$$\Rightarrow \frac{F(x) - F(a)}{G(x) - G(a)} = \frac{F(x)}{G(x)} = \frac{F(x)}{(x-a)^n}$$

(by GMVT) $\parallel$ (by GMVT) (by GMVT) (by GMVT)

$$\frac{F'(c_1)}{G'(c_1)} = \frac{F'(c_1) - F'(a)}{G'(c_1) - G'(a)} = \frac{F''(c_2)}{G''(c_2)} = \frac{F''(c_2) - F''(a)}{G''(c_2) - G''(a)} = \frac{F'''(c_3)}{G'''(c_3)} = \cdots = \frac{F^{(n)}(c_n)}{G^{(n)}(c_n)} = \frac{f^{(n)}(c_n)}{n!}$$

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Limit of Functions

Def: [Pointwise Convergence]

Def: [Uniform Convergence]

Th'm $C^0([a,b])$ is complete in sup. metric.

Example 1: On $\mathbb{R}$, $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - + \cdots$

Consider $f_1(x) = x$, $f_2(x) = f_1(x)$, $f_3(x) = x - \frac{1}{3!}x^3$, $f_4(x) = f_3(x)$, $\cdots$

Claim: $f_n(x) \longrightarrow \sin x$ uniformly on $[-M, M]$, $M < 1$.

(pf) $\left|\sin x - f_n(x)\right| = \left|\frac{1}{n!} x^n f^{(n)}(c)\right| \le \frac{1}{n!} M^n \longrightarrow 0$ #

Example 2: $f(x) = e^{-\frac{1}{x^2}}$

$f(0) = 0$, $f'(0) = \frac{1}{2x} e^{-\frac{1}{x^2}}\Big|_{x=0} = 0$, $f''(0) = \frac{1}{2x^2} e^{-\frac{1}{x^2}} + \frac{1}{4x^2} e^{-\frac{1}{x^2}}\Big|_{x=0} = 0$, $\cdots$