Online Math Camp (23S)

TA Session Note (5/22)

Dirichlet Function

$$f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \end{cases}, \quad p, q \in \mathbb{Z}, \quad (p, q) = 1.$$

1. $f(x)$ is continuous on $\mathbb{Q}$!

   Take $\frac{p}{q} \in \mathbb{Q}$, $\forall \delta > 0$, $\exists \varepsilon = \frac{1}{2q} \Rightarrow \cdots$

2. $f(x)$ is not continuous on $\mathbb{R} \setminus \mathbb{Q}$:

   Take $x \in \mathbb{R} \setminus \mathbb{Q}$, $\varepsilon > 0$, $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$.

   Let $d = \min \left\{ \left| x - \frac{p}{q} \right| \;\middle|\; q \le n,\ q \in \mathbb{N},\ p \in \mathbb{Z} \right\} > 0$

   Consider $\delta = d$, $\cdots$

Notation: $\begin{cases} f \nearrow \text{(increasing)}, & f \nearrow\!\!\nearrow \text{(strictly increasing)} \\ f \searrow \text{(de-\quad)} & f \searrow\!\!\searrow \text{(strictly de-\quad)} \end{cases}$

Monotone Function

Prop. $f(x^-)$, $f(x^+)$ exists for every $x \in \mathbb{R}$ if $f : \mathbb{R} \to \mathbb{R}$ is monotone.

pf) WLOG : $f \nearrow$.

<u>Claim</u> $f(x^-) = \sup\limits_{x'<x} f(x')$.

① $\forall \varepsilon > 0$, $\sup\limits_{x'<x} f(x') - \varepsilon$ is not an upper bound.

$$\Rightarrow \exists\, \delta > 0,\ f(x-\delta) > \sup_{x'<x} f(x') - \varepsilon$$

② $\forall\, \delta' < \delta$, $f(x-\delta') \geq f(x-\delta) > \sup\limits_{x'<x} f(x') - \varepsilon$
$(0<)$

Combine ①, ②, $\forall \varepsilon > 0$, $\exists\, \delta > 0$ such that

$$\delta' < \delta \Rightarrow \left| f(x) - \delta' \right| - \sup_{x'<x} f(x') < \varepsilon \quad \#$$

Prop. A monotone function has at most countably infinite discontinuities.

(pf) Label every discontinuity x with $(f(x^-), f(x^+))$.

The set of discontinuities gives you a set of disjoint open intervals.

(Think about the connection between open intervals and rationals in them.)

...

Differentiation

Def: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

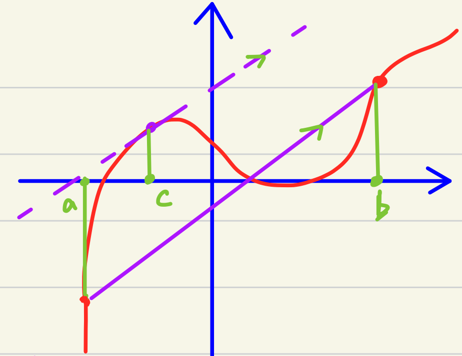
EX: $\frac{d}{dx} x^2 = 2x$:

(pf) $\frac{d}{dx} x^2 = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} (2x+h) = 2x$

$\forall \varepsilon > 0$, pick $\delta = \varepsilon$, then

$|h-0| < \delta \Rightarrow |h| < \delta \Rightarrow |(2x+h) - 2x| < \delta = \varepsilon$ #

MVT (Mean Value Theorem)



f: diff. $a, b \in \mathbb{R}$, $a < b$.

$\exists\, c \in (a, b)$ such that $\dfrac{f(b) - f(a)}{b - a} = f'(c)$.

(pf) Do Rolle's Thm on $g(x) = f(x) - (x-a) \cdot \dfrac{[f(b) - f(a)]}{b - a}$:

$g(a) = g(b) = f(a) \Rightarrow \exists\, c \in (a, b),\ g'(c) = 0\ \ldots$

General MVT

f, g : diff. $\quad g' \neq 0$, $g(a) \neq g(b)$.

$\Rightarrow \exists\, c \in (a,b)$, $\dfrac{f(a)-f(b)}{g(a)-g(b)} = \dfrac{f'(c)}{g'(c)}$

(pf) To show that $[f(a)-f(b)]\cdot g'(c) - [g(a)-g(b)]\cdot f'(c) = 0$,

we define $h(x) = [f(a)-f(b)]\,g(x) - [g(a)-g(b)]\,f(x)$

Since $h(a) = h(b)$, by Rolle's Thm, $\exists\, c$ such that $h'(c) = 0$ #.

Preview

Def: Let $\{f_n\}$ be a real-valued function defined on E.

f_n converges pointwisely to f if $\forall x \in E$, $\lim_{n\to\infty} f_n(x) = f(x)$.

denoted as $f_n \to f$.

Def: If f is bounded, $\|f\| = \sup_{x\in E} |f(x)|$

Def: We say f_n converges uniformly to f if

$\forall \varepsilon > 0$, $\exists N > 0$ such that $\forall n > N$, $\|f_n - f\| < \varepsilon$.

denoted as $f_n \xrightarrow{u} f$.

Prop. Let $C_b(E) = \{ f : E \to \mathbb{R}$ such that f is bounded and continuous$\}$
Then $C_b(E)$ is a metric space with metric $d(f, g) = \| f - g \|$.

Thm $f_n \xrightarrow{u} f$, f_n is bounded and continuous.
Then, f is bounded and continuous.

(pf) $|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)|$

Since $f_n \xrightarrow{u} f$, $\exists N > 0$ such that $\|f_n - f\| < \frac{\varepsilon}{3}$.

f is continuous $\Rightarrow \forall \varepsilon > 0$, $\exists \delta > 0$ such that $|f_N(x) - f_N(y)| < \frac{\varepsilon}{3}$ when $|x-y| < \delta$

$\Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$. i.e. f is continuous.

Also, $|f(x)| < |f_n(x) - f(x)| + |f_n(x)|$

$\Rightarrow \|f\| \leq \|f_n - f\| + \|f_n\| < M + \varepsilon \quad \Rightarrow f$ is bounded. #

Thm' $f_n \xrightarrow{u} f \iff \{f_n\}$ is Cauchy in $C_b(E)$.

(pf) ($\Rightarrow$) $\|f_n - f_m\| \leq \|f_n - f\| + \|f - f_m\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ for some $n, m > N$.

($\Leftarrow$) $\forall x \in E$, $|f_n(x) - f_m(x)| < \varepsilon$, $\forall n, m > N$

So $f_n(x)$ is a Cauchy sequence in $\mathbb{R}$.

Since $\mathbb{R}$ is complete, $f_n(x) \to f(x)$ in $\mathbb{R}$. $\Rightarrow f_n \to f$ (pointwisely)

$\forall x \in E$, $\exists\, m(x) \geq N$ such that $|f_{m(x)}(x) - f(x)| < \varepsilon$.

$$|f(x) - f_n(x)| \leq |f(x) - f_{m(x)}(x)| + |f_{m(x)}(x) - f_n(x)|, \quad \forall n > N.$$

$$< \varepsilon + \varepsilon = 2\varepsilon$$

Take $\varepsilon' = 2\varepsilon$. done. #

Thm There exists a continuous function on $\mathbb{R}$ that is nowhere differentiable.

Ex: $f_1 =$  f_1

$f_2 =$  $f_1 + f_2$

$f_3 =$  $f_1 + f_2 + f_3$

$\vdots$ $\qquad\qquad$ $\vdots$

(Chp. 9 of Rudin)

Construction $\varphi(x) = |x|, \quad -1 \leq x \leq 1, \quad \varphi(x+2) = \varphi(x)$

Let $$f(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \varphi(4^n x)$$

Then f is continuous, but nowhere diff.