Online Math Camp (23S)

TA Session Note (5/15)

Continuity

Def: $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\underbrace{|x'-x| < \delta}_{x' \in B_\delta(x)} \Rightarrow \underbrace{|f(x') - f(x)| < \varepsilon}_{f(x') \in B_\varepsilon(f(x))}$.

Th'm $f: X \to Y$ continuous $\Leftrightarrow$ $\forall V$ open, $V \subseteq Y$, $\exists U$ open, $U \subseteq X$ such that

$$x \in U \Rightarrow f(x) \in V$$

Th'm $f: X \to Y$ continuous $\Leftrightarrow$ $\forall V$ open, $V \subseteq Y$ $\Rightarrow$ $f^{-1}(V)$ open, $f^{-1}(V) \subseteq X$.

(pf) ($\Rightarrow$) Want to Show (WTS) $\forall x \in f^{-1}(V)$, $\exists r > 0$ such that $\underline{B_r(x) \subseteq f^{-1}(V)}$

$x' \in B_r(x) \Rightarrow x' \in f^{-1}(V)$

$\Updownarrow \qquad\qquad \Updownarrow$

$d(x', x) < r \qquad f(x') \in V$

Take $x \in f^{-1}(V)$

V open $\Rightarrow \exists r' > 0$ such that $B_{r'}(f(x)) \subseteq V$

$\Rightarrow \exists r > 0$ such that $B_r(x) \subseteq f^{-1}(B_{r'}(f(x))) \subseteq f^{-1}(V)$ #

($\Leftarrow$) Take V as open ball. #

Corr. $f^{-1}(F)$ is closed if F is closed and f is continuous.

(pf) $f^{-1}(F) = \Big(\underbrace{f^{-1}(\underbrace{F^c}_{\text{open}})}_{\text{open by Thm}}\Big)^c$ is closed since $f^{-1}(F^c)$ is open. #

Thm $f : X \to Y$ is continous and $g : Y \to Z$ is continuous.

$\Rightarrow g\circ f : X \to Z$ is continuous.

(pf #1) U open, $U \subseteq Z \Rightarrow (g\circ f)^{-1}(U) = \underbrace{f^{-1}(\underbrace{g^{-1}(U)}_{\text{open by Thm}})}_{\text{open by Thm}}$ is open.

By above Thm, $g\circ f$ is continuous. #

(pf #2) WTS: $\forall \varepsilon > 0 \ \exists \delta > 0$ such that $B^X_\delta(x) \subseteq (g\circ f)^{-1}\left(B^Z_\varepsilon(g\circ f(x))\right)$

Since g is continuous, $\exists \delta_1 > 0$ such that $B^Y_{\delta_1}(f(x)) \subseteq g^{-1}\left(B^Z_\varepsilon((g\circ f)(x))\right)$

f is continuous, $\exists \delta_2 > 0$ such that $B^X_{\delta_2}(x) \subseteq f^{-1}\left(B^Y_{\delta_1}(f(x))\right)$

$\subseteq f^{-1}\left(g^{-1}\left(B^Z_\varepsilon((g\circ f)(x))\right)\right)$. #

Th'm $f: X \to Y$, continuous, K compact, $K \subseteq X$. $\Rightarrow f(K)$ compact & $f(K) \subseteq Y$.

(pf) $\{V_\alpha\}$: open cover of $f(K)$.

$\Rightarrow \{f^{-1}(V_\alpha)\}$: open cover of K.

$\Rightarrow \{f^{-1}(V_{\alpha_i})\}$: finite open cover of K.

$\Rightarrow \{V_{\alpha_i}\}$: finite open cover of $f(K)$. #

Uniformly Continuous

Th'm $f: X \to Y$ continuous, X: compact $\Rightarrow f$ is uniformly continuous.

i.e. $\forall \varepsilon > 0, \exists \delta > 0, \underline{\forall x \in X}, \; d(x', x) < \delta \Rightarrow d(f(x'), f(x)) < \varepsilon$

Independent of x !!

Thm $f: X \to Y$, continuous. C connected in $X \Rightarrow f(C)$ connected in Y.

(pf) Suppose $f(C) = A \cup B$, non-empty & separated.

<u>Claim:</u> $f^{-1}(A)$, $f^{-1}(B)$ is separated.

i.e. $f^{-1}(A) \cap f^{-1}(B) \neq \phi$

$$\subseteq \quad f^{-1}(\overline{A}) \cap f^{-1}(B) = f^{-1}(\overline{A} \cup B)$$

\#

IVT $f: X \longrightarrow \mathbb{R}$ continuous, X connected

If $f(x_1) = a$, $f(x_2) = b$, $c \in [a, b]$,

then $\exists\, x \in X$ such that $f(x) = c$.

Differentiability

Def: A function $f:[a,b]\to\mathbb{R}$ is differentiable at x if

$$\lim_{t\to x}\frac{f(x)-f(t)}{x-t}=L,$$

and the derivative of f at x is $f'(x)=L$.

Def: A function $f:[a,b]\to\mathbb{R}$ is differentiable if it is differentiable at all $x\in[a,b]$.

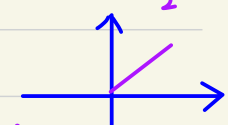
Ex: $f(x)=x^2$:

$$\lim_{t\to x}\frac{f(x)-f(t)}{x-t}=\lim_{t\to x}\frac{x^2-t^2}{x-t}=\lim_{t\to x}x+t=2x,\ \Rightarrow f'(x)=2x.$$

Ex: $f(x)=\sin x$:

$$\lim_{t\to x}\frac{\sin x-\sin t}{x-t}=\lim_{t\to x}\frac{2\cos\left(\frac{x+t}{2}\right)\sin\left(\frac{x-t}{2}\right)}{x-t}=\lim_{t\to x}\cos\left(\frac{x+t}{2}\right)\cdot\frac{\sin\left(\frac{x-t}{2}\right)}{\frac{x-t}{2}}$$

$$=\cos x=f'(x)$$



Ex: $f(x)=|x|$ is not diff. at $x=0$:

$$\lim_{t\to 0}\frac{|t|-|0|}{t-0}=\lim_{t\to 0}\frac{|t|}{t}\text{ doesn't exist.}$$

Prop. f is diff $\Rightarrow$ f is continous

(pf) If $\lim_{t \to x} \frac{f(x)-f(t)}{x-t}$ exist, then $x \to t \to 0$ $\Rightarrow$ $f(x)-f(t) \to 0$. #

Note: The converse is not true. Continuous function $f(x)=|x|$ is not diff. at $x=0$.

Ex: If f is diff, is f' continuous?

No! $f(x)=\begin{cases} \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x=0 \end{cases}$ is diff. $\Rightarrow$ $f'(x)=\begin{cases} 2x\sin\left(\frac{1}{x}\right)-\cos\left(\frac{1}{x}\right), & x\neq 0 \\ 0, & x=0 \end{cases}$

Not continuous at $x=0$

Prop. (i) $(f+g)' = f' + g'$

(ii) $(fg)' = f'g + fg'$

(iii) $f = c \Rightarrow f' = 0$

(iv) $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

(pf) (i) $\lim_{t \to x} \frac{f(x)+g(x)-f(t)-g(t)}{x-t}$

$= \lim_{t \to x} \frac{f(x)-f(t)}{x-t} + \lim_{t \to x} \frac{g(x)-g(t)}{x-t}$

$= f'(x) + g'(x)$ #

(pf) (ii) $\lim_{t \to x} \frac{f(x)g(x) - f(t)g(t)}{x-t} = \lim_{t \to x} \frac{f(x)g(x) - f(x)g(t) + f(x)g(t) - f(t)g(t)}{x-t}$

$= \lim_{t \to x} \left[f(x) \frac{g(x)-g(t)}{x-t} + g(t) \cdot \frac{f(x)-f(t)}{x-t} \right] = f(x)g'(x) + f'(x)g(x)$ #

(iii) $\lim_{t \to x} \frac{c-c}{x-t} = 0$ #

(iv) Since $\left(g \cdot \frac{1}{g}\right)' = 0 = g' \cdot \frac{1}{g} + g\left(\frac{1}{g}\right)' \Rightarrow \left(\frac{1}{g}\right)' = -\frac{g'}{g^2}$

By (ii), $\left(\frac{f}{g}\right)' = f' \cdot \frac{1}{g} + f\left(\frac{1}{g}\right)' = \frac{f'g - f \cdot g'}{g^2}$. #

Ex: $f(x) = x^2$. $\Rightarrow f'(x) = (x)' \cdot x + x \cdot (x)' = 2x$ since $(x)' = \lim_{t \to x} \frac{x-t}{x-t} = 1$.

$\Rightarrow (x^3)' = (x^2)' \cdot x + x^2 \cdot (x)' = 2x^2 + x^2 = 3x^2$

By Induction, $(x^n)' = n x^{n-1}$.

Ex: $f(x) = \frac{\sin x}{x}$, $x \neq 0$ $\Rightarrow f'(x) = \frac{1}{x^2}\left[(\sin x)' x - (\sin x) \cdot (x)'\right] = \frac{(\cos x) \cdot x - \sin x}{x^2}$

for $x \neq 0$.

Rolle's Thm If $f : [a, b] \longrightarrow \mathbb{R}$ is diff. and $f(a) = f(b)$,

Then, $\exists\, c \in (a, b)$, such that $f'(c) = 0$.

(pf) If $f(x) = c$, then $f'(x) = 0$ $\forall x \in [a, b]$

If not, by Weierstrauss Thm, f achieves maximum or minimum is (a, b).

Suppose $f(c)$ is maximum (WLOG), $c \in (a, b)$,

$$f'(c) = \lim_{t \to c} \frac{f(c) - f(t)}{c - t} = 0 \quad \text{since} \begin{cases} \lim_{t \to c^+} \frac{\overbrace{f(c) - f(t)}^{\geq 0}}{\underbrace{c - t}_{< 0}} \leq 0 \\ \lim_{t \to c^-} \frac{\overbrace{f(c) - f(t)}^{\geq 0}}{\underbrace{c - t}_{> 0}} \geq 0 \end{cases}$$

#

MVT (Mean Value Thm)

If $f : [a,b] \longrightarrow \mathbb{R}$ is diff., then $\exists\ c \in (a,b)$ such that $f(b) - f(a) = f'(c)(b-a)$.

(pf) $g(x) = f(x) - rx$, $r = \dfrac{f(b)-f(a)}{b-a}$

$g(b) - g(a) = [f(b) - f(a)] - r(b-a) = 0 \Rightarrow g(b) = g(a)$.

By Rolle's Thm, $\exists\ c \in (a,b)$ such that $g'(c) = f'(c) - r = 0$

$\Rightarrow f'(c) - r = \dfrac{f(b)-f(a)}{b-a}$. #

Prop. If $f'(x) > 0$, $\forall x$, then f is increasing.

(pf) For $b > a$, $f(b) - f(a) = \underbrace{f'(c)}_{>0} \cdot (b-a)$, for some $c \in (a,b)$

So, $f'(c) > 0 \Rightarrow f(b) > f(a)$. #