Online Math Camp (23S)

TA Session Note (5/8)

Thm The series $\underbrace{1+\frac{1}{2}+\frac{1}{3}+\cdots}_{\text{Harmonic Series}}=\sum\frac{1}{n}$ diverges

(pf) $1+\frac{1}{2}+\underline{\frac{1}{3}+\frac{1}{4}}+\underline{\frac{1}{5}+\cdots+\frac{1}{8}}+\underline{\frac{1}{9}+\cdots+\frac{1}{16}}+\cdots$

$\geq 1+\frac{1}{2}+\underbrace{\frac{1}{4}+\frac{1}{4}}_{=\frac{1}{2}}+\underbrace{\frac{1}{8}+\cdots+\frac{1}{8}}_{=\frac{1}{2}}+\underbrace{\frac{1}{16}+\cdots+\frac{1}{16}}_{=\frac{1}{2}}+\cdots$ which diverges. #

Comparison Test

Let $a_n \geq b_n \geq 0$. $\Rightarrow \begin{cases} \text{① If } \sum a_n \text{ converges, then } \sum b_n \text{ converges.} \\ \text{② If } \sum b_n \text{ diverges, then } \sum a_n \text{ diverges.} \end{cases}$

Root Test

Let $a_n \geq 0$. $\Rightarrow \begin{cases} \text{① If } \limsup \sqrt[n]{a_n} < 1, \text{ then } \sum a_n \text{ converges.} \\ \text{② If } \limsup \sqrt[n]{a_n} > 1, \text{ then } \sum a_n \text{ diverges.} \end{cases}$

(Sketch of pf)

① $\limsup \sqrt[n]{a} < 1 \Rightarrow \exists\, \beta < 1,\ N \in \mathbb{N}$ such that $\sqrt[n]{a_n} < \beta$ $\left(\text{e.g. } \beta = \frac{\limsup \sqrt[n]{n} + 1}{2}\right)$

$\Rightarrow a_n < \beta^n$. By comparison test $\sum a_n < \sum \beta^n$ converges. #

② $\forall N \in \mathbb{N},\ \exists\, n \in \mathbb{N}$ such that $n > N,\ a_n > 1 \Rightarrow a_n \not\to 0$ #

Ratio Test

Let $\sum a_n$ be a series, and $a_n \neq 0$ for large enough n.

① If $\limsup \left| \frac{a_{n+1}}{a_n} \right| < 1$, then $\sum a_n$ converges.

② If $\left| \frac{a_{n+1}}{a_n} \right| \geq 1$ for n large enough, then $\sum a_n$ diverges.

Abel's Thm

$A_n = \sum_{i=0}^{n} a_i$ is bounded, b_n is monotonic decreasing, and $b_n \to 0$.

Then, $\sum a_n b_n$ converges.

EX: $\sum \underbrace{\frac{1}{n}}_{b_n} \underbrace{\sin\left(\frac{n\pi}{4}\right)}_{a_n}$ converges by Abel's Thm.

Absolute Convergence

Does $1-1+\frac{1}{2}-\frac{1}{2}+\frac{1}{2}-\frac{1}{2}+\frac{1}{3}-\frac{1}{3}+\frac{1}{3}-\frac{1}{3}+\frac{1}{3}-\frac{1}{3}+\cdots$ converge?

$$\underbrace{1-1}_{=0}+\underbrace{\frac{1}{2}-\frac{1}{2}}_{=0}+\underbrace{\frac{1}{2}-\frac{1}{2}}_{=0}+\underbrace{\frac{1}{3}-\frac{1}{3}}_{=0}+\underbrace{\frac{1}{3}-\frac{1}{3}}_{=0}+\underbrace{\frac{1}{3}-\frac{1}{3}}_{=0}+\cdots \to 0$$

But $1-1+\underbrace{\frac{1}{2}+\frac{1}{2}}_{1}\underbrace{-\frac{1}{2}-\frac{1}{2}}_{-1}+\underbrace{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}}_{1}\underbrace{-\frac{1}{3}-\frac{1}{3}-\frac{1}{3}}_{-1}+\cdots$

fluctuates between $1, 0, 1, 0, \ldots$

Def: Let $\sum a_n$ be a convergent sequence.

If $\sum |a_n|$ also converges, we say that $\sum a_n$ converges **absolutely**.

If $\sum |a_n|$ doesn't converges, we say that $\sum a_n$ converges **conditionally**.

Rearrangement

$\sum a_i$ converges absolutely, so $\exists N \in \mathbb{N}$ such that $n > N \Rightarrow \sum_{i=n}^{\infty} |a_i| < \varepsilon$.

$\sum a'_{\sigma(i)}$ is a rearrangement of $\sum a_i \Rightarrow \left| \sum_{j=n}^{m} a'_j \right| < \varepsilon$ for n large enough

(pf) Let $S = \{\sigma(i) \mid i = 1, \cdots, N\}$. Take $M = \max S + 1$

Then $n > M \Rightarrow \sum_{\sigma(i)=M}^{\infty} \left| a'_{\sigma(i)} \right| < \sum_{n=N}^{\infty} |a_i| < \varepsilon$ #

Product

$\sum a_n$, $\sum b_n$ converges absolutely, $\quad c_n = \sum_{i=0}^{n} a_i b_{n-i}$, $\Rightarrow \sum c_n$ converges.

(pf)

$$\begin{array}{ccccc}
a_1 b_1 & a_2 b_1 & a_3 b_1 & a_4 b_1 & \cdots \\
a_1 b_2 & a_2 b_2 & a_3 b_2 & a_4 b_2 & \\
a_1 b_3 & a_2 b_3 & a_3 b_3 & a_4 b_3 & \\
a_1 b_4 & a_2 b_4 & a_3 b_4 & a_4 b_4 & \\
\vdots & & & &
\end{array}$$

$+ \quad a_2 b_3$

$\Rightarrow$ ok.

Limit of Function

Def:

Prop. Limit of Sequence meets limit of function

$$\lim_{x \to p} f(x) = L \iff \forall x_n \underset{\neq p}{\longrightarrow} p,\ f(x_n) \longrightarrow L.$$

Prop. If f is continuous, then $f\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} f(x_n)$ as $x_n \to p$

Collarary: If f, g are continuous, then $f+g$, $f \cdot g$ are continuous,
and f/g is continuous if $g \neq 0$ at all x.

Cor $f: X \to Y$ is continuous $\Leftrightarrow$ $\forall$ closed K in Y, $f^{-1}(K)$ is closed in X

pf $\forall$ open U in X, U^c is closed, $f^{-1}(U^c)$ is closed

$\Rightarrow (f^{-1}(U^c))^c$ is open $\Rightarrow f^{-1}(U)$ is open

Similar for another direction. #

Thm If $f: X \to Y$ is conti. then for any compact K in X, $f(K)$ is compact in Y

pf Let $\{U_\alpha\}$ be an open cover of $f(K)$ in Y.

So $\{f^{-1}(U_\alpha)\}$ is an open cover of K

Since K is compact, $\exists$ finite subcover $f^{-1}(U_1), \dots, f^{-1}(U_n) \Rightarrow U_1, \dots, U_n$ is a finite subcover of $f(K) \Rightarrow f(K)$ is compact. #

Homeomorphism thm: If $f: X \to Y$ is conti bijective on a compact set X, then f is a homeomorphism, i.e. f^{-1} is continuous.

pf. f^{-1} is continuous $\Leftrightarrow$ $\forall$ closed K in X, $f(K)$ is closed in Y.

$\forall$ closed K in X, K is compact. Since f is conti, $f(K)$ is compact $\Rightarrow$ $f(K)$ is closed. ※

Cor. $[0,1]$ is not homeomorphic to $\mathbb{R}^1$.

Weierstrass thm. If $f: X \to \mathbb{R}$ and f is conti.,

X is compact. Then f achieves it maximum and minimum.

pf. Since f is conti., X is compact, $f(X)$ is compact

in $\mathbb{R}$. By Heien-Borel, $f(X)$ is closed and bounded

Bounded $\Rightarrow$ $\sup f$ and $\inf f$ exist

Closed $\Rightarrow$ $\exists x, y \in X$ s.t. $f(x) = \sup(f)$, $f(y) = \inf(f)$

Example in Economics: Consumer Theory

Suppose in a market, there are n goods, $x_1, \cdots, x_n$ with price $p_1, \cdots, p_n > 0$.

Every individual maximizes its utility under the budget constraint:

$$\max_{x_1,\cdots,x_n} u(x) \quad \text{s.t.} \quad \sum p_i x_i \leq y$$

If $u(\cdot)$ is continuous, then the maximization point exists, and the best consumption bundle is called the Mashallian demand.

(pf) Let $A = \{(x_1, \cdots, x_n) \mid 0 \leq \sum p_i x_i \leq y,\ x_i \geq 0\}$.

A is bounded: A is bounded by $[0, \frac{y}{p_1}] \times \cdots \times [0, \frac{y}{p_n}]$.

A is closed: Since $f = \sum p_i x_i$ is continuous, $f^{-1}([0, y]) = A$ is closed.

Hence, by Heien-Borel Th'm, A is compact.

By Weierstrass Th'm, u achieves its maximum. #