Online Math Camp (23S)

TA Session Note (5/1)

Cauchy Sequence

EX: $x_1 = 1$, $x_2 = 2$, $x_n = \frac{1}{2}(x_{n-1} + x_{n-2})$ is Cauchy.

(pf) $|x_1 - x_2| = 1$, $|x_2 - x_3| = \frac{1}{2}$, $|x_3 - x_4| = \frac{1}{4}$, ...

In general, $|x_n - x_{n-1}| = \left|\frac{1}{2}(x_{n-1} + x_{n-2}) - x_{n-1}\right| = \frac{1}{2}|x_{n-2} - x_{n-1}|$

Hence, $|x_n - x_{n-1}| = \frac{1}{2^{n-1}}$, and for $m > n$,

$$|x_n - x_m| \le |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \cdots + |x_{m-1} - x_m|$$

$$= \sum_{i=1}^{m-1} |x_i - x_{i+1}| = \sum_{i=n}^{m-1} \frac{1}{2^{i-1}} \le \sum_{i=n}^{\infty} \frac{1}{2^{i-1}} = \frac{1}{2^{n-2}}$$

(Use ε-δ argument to finish...)

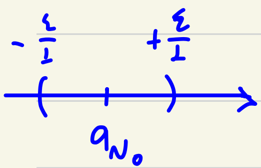
Complete

Def: X is complete iff for any $\{a_n\}$, Cauchy sequence in X $\Rightarrow$ $\{a_n\}$ converges.

Prop. $\mathbb{R}$ is complete.

(pf) Take $\{a_n\}$: Cauchy. $\{a_n\}$ converges if $\limsup a_n = \liminf a_n$.

<u>Claim</u>: $|\limsup a_n - \liminf a_n| < \varepsilon \quad \forall \varepsilon > 0$ (if $n > N(\varepsilon)$).



$\forall \varepsilon > 0$, find $N_0 \in \mathbb{N}$ such that if $m, n > N_0$, then $|a_m - a_n| < \frac{\varepsilon}{2}$.

$$\Rightarrow \sup_{n > N_0} a_n \leq a_{N_0} + \frac{\varepsilon}{2}, \quad \inf_{n > N_0} a_n \geq a_{N_0} - \frac{\varepsilon}{2}$$

$$\Rightarrow |\limsup a_n - \liminf a_n| \leq \left|\sup_{n > N_0} a_n - \inf_{n > N_0} a_n\right| \leq \left|\left(\cancel{a_{N_0}} + \frac{\varepsilon}{2}\right) - \left(\cancel{a_{N_0}} - \frac{\varepsilon}{2}\right)\right|$$

$$\leq \varepsilon \quad \#$$

Examples:

$$\begin{cases} \limsup (-1)^n = ? \\ \liminf (-1)^n = ? \end{cases} \qquad (-1)^n = \begin{cases} 1, & \text{if } n \text{ even} \\ -1, & \text{if } n \text{ odd.} \end{cases}$$

Since $\begin{cases} \limsup (-1)^n = \max \{ \limsup(1), \limsup(-1) \} \\ \liminf (-1)^n = \min \{ \liminf(1), \liminf(-1) \} \end{cases}$, then are easy: $\begin{cases} = 1 \\ = -1. \end{cases}$

Series

Def: Given a sequence $\{a_n\}$, we say a series $\sum_{n=1}^{\infty} a_n$ converges if the sequence $\left\{\sum_{i=1}^{n} a_i\right\}$ converges.

In the case, $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$

Cauchy Criterion

$\forall \varepsilon > 0$, $\exists N_0 \in \mathbb{N}$, such that $n > m > N_0$, $\Rightarrow \underbrace{\left| \sum_{i=1}^{n} a_i - \sum_{i=1}^{m} a_i \right|}_{\left| \sum_{i=m+1}^{n} a_i \right|} < \varepsilon$

Example: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = ?$

Trick: Find a, b so that $\displaystyle\frac{1}{n(n+1)} = \frac{a}{n} + \frac{b}{n+1} = \frac{(a+b)n+1}{n(n+1)} \Rightarrow a=1,\ b=-1$

Here, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cancel{\cdots} = 1.$

Limit-Sup:

$$\limsup a_n = \max\{\text{limit of subsequence}\}$$

Prop. $\limsup\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\left(\sup\limits_{k>n} a_n\right) = \inf\left(\sup\limits_{k>n} a_n\right).$

(Sketch of Proof)

1. $\exists\, a_{p_n} \to \limsup a_n$

$\forall \varepsilon>0$, $\exists\, n\in\mathbb{N}$ such that $\sup\limits_{k>n} a_n \in \left(\inf\left(\sup\limits_{k>n} a_n\right), \inf\left(\sup a_n\right)+\varepsilon\right).$

$\Rightarrow \exists\, k>n$ such that $a_k \in \left(\inf(\sup a_n), \sup(a_n)\right)$

$\Rightarrow |a_k - \limsup a_n| < \varepsilon.$

2. If $L > \limsup$, then $\nexists\, a_{p_n} \to L$.

(pf) $\exists\, n \in \mathbb{N}$, such that $\sup\limits_{k>n} a_n < L$.

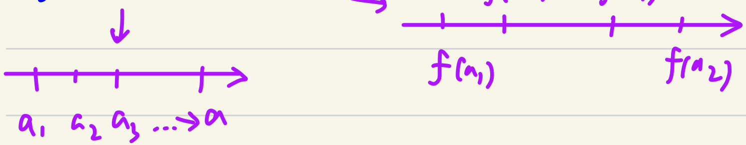
$\Rightarrow |a_k - L| > |\sup a_n - L| \quad \forall k > n \quad \Rightarrow L$ cannot be the limit.

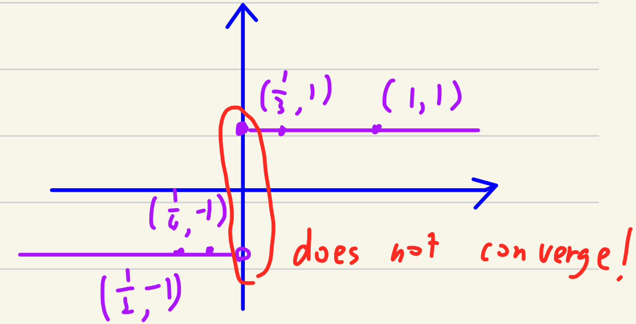
3. Inf (same).

Preview of Next Week

Limit of Functions & Continuity

$f: M \longrightarrow N$





$$f(x) = \begin{cases} 1 & , \ x \geq 0 \\ -1 & , \ x < 0 \end{cases}$$

for $a_1 = 1, \ a_2 = -\frac{1}{2}, \ a_3 = \frac{1}{3}, \ a_4 = -\frac{1}{4}, \ \ldots$

$f(a_1) = 1, \ f(a_2) = -1, \ f(a_3) = 1, \ -(a_4) = -1, \ \ldots$

Def: $\lim\limits_{x \to p} f(x) = q$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $d_M(x, p) < \delta \Rightarrow d_N(f(x), q) < \varepsilon$

EX: $f(x) = x$: $\lim\limits_{x \to 2} f(x) = 2$.

(pf) $\forall \varepsilon > 0$, let $\delta = \varepsilon$. then $|x - 2| < \delta \Rightarrow |f(x) - 2| = |x - 2| < \delta = \varepsilon$ #

Def: f is continuous at p if

$\forall \varepsilon > 0$, $\exists \delta > 0$, such that $d_M(x, p) < \delta \Rightarrow d_N(f(x), q) < \varepsilon$.

Def 1: f is continuous on M if f is continous at every point $p \in M$.

Def 2: f is continuous on M if f preserves every convergence sequence, so that the image of convergence sequences in M is a convergence sequence in N.

Thm: Def. 1 $\Longleftrightarrow$ Def. 2

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(pf) <u>Suppose Def. 1 holds.</u> $\forall \{a_n\} \to a$ in M $\overset{?}{\Rightarrow} f(a_n) \to f(a)$

Since f is continuous, f is continuous at a

$\forall \varepsilon > 0$, $\exists\, \delta > 0$ such that $d_M(x, a) < \delta \Rightarrow d_N(f(x), f(a)) < \varepsilon$.

Since $\{a_n\}$ converges, $\exists\, N > 0$ such that $\forall n > N$,

$$d_M(a_n, a) < \delta \Rightarrow d_N(f(a_n), f(a)) < \varepsilon$$

Hence, $\{f(a_n)\} \to f(a)$, i.e. Def. 2 holds.

<u>Suppose Def. 2 holds.</u>

If Def. 1 does not hold, then $\exists\, p \in M$ such that f is not continuous at p.

i.e. $\exists\, \varepsilon > 0$ such that $\forall \delta > 0$, $\exists\, x$ such that $d_M(x, p) < \delta$, but $d_N(f(x), f(p)) \geq \varepsilon$.

Let $\delta_n = \frac{1}{n}$, $\exists\, x_n$ such that $d_M(x_n, p) < \frac{1}{n}$, but $d_N(f(x_n), f(p)) \geq \varepsilon$. $\Rightarrow \{x_n\} \to p$.

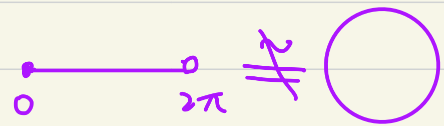
However, $d_N(f(x_n), f(p)) \geq \varepsilon > 0$ $\forall n \in \mathbb{N}$ $(\rightarrow\leftarrow)$

Homeomorphism

Def: $f: M \to N$ is a homeomorphism if f is continuous, bijective, and f^{-1} is also continuous.

Def: If there exists a homeomorphism f between M and N, we says M is homeomorphic to N, or $M \cong N$.

EX: $Y \cong T$.



EX: A coffee cup is homeomorphic to a donut.