Online Math Camp (2023)

TA Session Note (4/23)

Sequences

Def: $\{p_n\}$ is a sequence, and $\lim_{n\to\infty} p_n = p$ mean

"$\forall \varepsilon>0$, $\exists N\in\mathbb{N}$ such that $n>N \Rightarrow d(p_n, p)<\varepsilon$."

Prop. 1: $\{p_n\}$ converges $\Rightarrow$ $\{p_n\}$ is bounded.

(pf) By def., $\exists N\in\mathbb{N}$ such that $n>N \Rightarrow d(p_n, p)<$ "1".

$\Rightarrow \{p_n\} \subseteq \{p_1, \cdots, p_N\} \cup [p-1, p+1]$, bounded $\Rightarrow$ $\{p_n\}$ is bounded. #

Prop. 2: $\lim_{n\to\infty} p_n = p$, $\lim_{n\to\infty} p_n = p'$, Then $p=p'$

(pf) Suppose $p\neq p'$, take $\underline{\varepsilon = \frac{1}{2}d(p,p')}$, then you can find $N_1, N_2 \in\mathbb{N}$

such that $\begin{cases} n>N_1 \Rightarrow d(p,p_n)<\varepsilon \\ n>N_2 \Rightarrow d(p',p_n)<\varepsilon \end{cases}$ $\Rightarrow$ For $N=\max\{N_1, N_2\}$, such that

$n>N \Rightarrow d(p,p_n), d(p',p_n)<\varepsilon$

Hence, $d(p,p') \overset{\text{(triangular iq.)}}{\leq} \varepsilon+\varepsilon = 2\varepsilon = \underline{d(p,p')}$ $(\rightarrow\leftarrow)$

Prop. 3: If E contains a limit point p.
Then $\exists \{p_n\} \in E$ such that $\lim_{n\to\infty} p_n = p$.

(pf) $\forall n \in \mathbb{N}$, pick p_n such that $d(p_n, p) < \frac{1}{n}$. $\Rightarrow \lim_{n\to\infty} p_n = p$. #

Arithmetic of Sequences

1. $\begin{cases} a_n \to a \\ b_n \to b \end{cases} \Rightarrow a_n + b_n \to a+b$

(pf) $\forall \varepsilon > 0$, $\exists N_1, N_2 \in \mathbb{N}$ such that $\begin{cases} n > N_1 \Rightarrow |a_n - a| < \frac{\varepsilon}{2} \\ n > N_2 \Rightarrow |b_n - b| < \frac{\varepsilon}{2} \end{cases}$

Hence, for $N = \max\{N_1, N_2\}$, $|(a_n + b_n) - (a+b)| \le |a_n - a| + |b_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ #

2. $\begin{cases} a_n \to a \\ b_n \to b \end{cases} \Rightarrow a_n - b_n \to a-b$

(pf) Use $|(a_n - b_n) - (a-b)| \le |a_n - a| + |b - b_n| < \varepsilon$ (triangular inequality)
(Same as 1.)

3. $a_n \to a$, c: constant $\Rightarrow c \cdot a_n \to c \cdot a$.

(pf) If $c = 0$, trivial.

If $c \neq 0$, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $|a_n - a| < \frac{\varepsilon}{|c|}$.

Then $|c \cdot a_n - c \cdot a| = |c| \cdot |a_n - a| < \varepsilon$ #

4. $\begin{cases} a_n \to a \\ b_n \to b \end{cases} \Rightarrow a_n \cdot b_n \to a \cdot b$

(pf) Use $|a_n b_n - ab| = |b_n(a_n - a) + a(b_n - b)| \le |b_n| \cdot |a_n - a| + |a| \cdot |b_n - b|$

$< B \cdot |a_n - a| + |a| \cdot |b_n - b|$.

$\{b_n\}$ converges $\Rightarrow |b_n| < B$ for some $B > 0$

$\forall \varepsilon > 0$, $\exists N_1, N_2 \in \mathbb{N}$ such that $\begin{cases} n > N_1 \Rightarrow |a_n - a| < \frac{\varepsilon}{2B} \\ n > N_2 \Rightarrow |b_n - b| < \frac{\varepsilon}{2(|a|+1)} \end{cases}$

Then, ...

5. $a_n \neq 0 \ \forall n \in \mathbb{N}$, $\{a_n\} \to a \neq 0$. $\Rightarrow \frac{1}{a_n} \to \frac{1}{a}$.

(pf) $\left|\frac{1}{a_n} - \frac{1}{a}\right| = \left|\frac{a-a_n}{a \cdot a_n}\right| \leq \left|\frac{1}{a}\right| \cdot \left|\frac{2}{a}\right| |a-a_n|$ since $\left|\frac{1}{a_n}\right| \leq \left|\frac{2}{a}\right| \Leftrightarrow |a_n| \geq \left|\frac{a}{2}\right|$

exists!

(true for $n \geq \underline{N_1}$)

$\exists N_1$ such that $|a_n - a| \leq \frac{|a|}{2} \Rightarrow |a_n| \leq |a_n - a| + |a| = \frac{3}{2}|a|$

...

Example:

1. $\frac{3n}{n+2} \longrightarrow 3$

$= \frac{3}{1+\frac{2}{n}}$

$\left(\text{since } n+2 > N+2 > \frac{6}{\varepsilon}+2 > \frac{6}{\varepsilon} \right)$

(pf) $\forall \varepsilon > 0$, $\exists N > \frac{6}{\varepsilon}$. such that $n > N \Rightarrow \left| \frac{3n}{n+2} - 3 \right| = \left| \frac{6}{n+2} \right| < \left| \frac{6}{\frac{6}{\varepsilon}} \right| = \varepsilon$ #

2. $\left(\frac{3n}{n+2} \right)^2 \longrightarrow 9$

(pf) Plan: Want to show $\left| \frac{9n^2}{n^2+4n+4} - 9 \right| = 9 \cdot \left| \frac{1}{1+\frac{4}{n}+\frac{4}{n^2}} - 1 \right| = 9 \cdot \left| \frac{\frac{4}{n}+\frac{4}{n^2}}{1+\frac{4}{n}+\frac{4}{n^2}} \right|$

$< 9 \cdot \left| \frac{4}{n} + \frac{4}{n^2} \right| \leq \left| \frac{8}{n} \right| < \varepsilon$ ...

(The rest is HW)

3. $p > 1 \Rightarrow p^{\frac{1}{n}} \to 1.$

(pf) Let $p' = p-1$, $\Rightarrow (1+p')^{\frac{1}{n}} \leq 1 + \frac{p'}{n}$ by Bernolli inequality.

4. $n^{\frac{1}{n}} \to 1.$

[Let $n^{\frac{1}{n}} = 1+k \Rightarrow (1+k)^n = 1+nk+\frac{n(n-1)}{2}k^2 + \cdots > \frac{n(n-1)}{2}k^2 > \frac{(n-1)^2}{2}k^2 > n.$]

(pf) $\forall \varepsilon > 0$, $\exists N > \frac{2}{\varepsilon^2}+1$, $(1+\varepsilon)^n > n$ if $n > N \Rightarrow |n^{\frac{1}{n}} - 1| < \varepsilon.$

Theorem $\{p_n\}$ is bounded & monotone $\Rightarrow$ $\{p_n\}$ converges

Prop. $\{a_n\}$ bounded in $\mathbb{R}$ $\Rightarrow$ $\exists$ sub-sequence $\{a_n'\} \subseteq \{a_n\}$ that converges

(pf) $\{a_n\}$ bounded $\Rightarrow$ $\{a_n\} \subseteq B_r(0)$ for some $r > 0$.

Since $B_r(0)$ is compact, $\{a_n\}$ has a limit point.

$\Rightarrow \exists \{a_n'\} \subseteq \{a_n\}$ that converges. #

Preview of Next Week

Monotone Convergence Theorem

Every bounded and monotone sequence converges to its limits.

If a_n is $\begin{cases} \text{increasing } (\nearrow), \\ \text{decreasing } (\searrow) \end{cases}$ $\{a_n\}$ converges to its $\begin{cases} \sup \\ \inf. \end{cases}$

(pf) Suppose $a_n \nearrow$.

Since $\{a_n\}$ is bounded & non-empty, $\sup a_n = a$ exists by l.u.b property.

i.e. $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $a - a_n < \varepsilon$ $\forall n > N$.

(Since otherwise $a-\varepsilon$ would be the true l.u.b.)

Hence, $|a - a_n| < \varepsilon$ $\forall n > N$ $\Rightarrow \lim\limits_{n\to\infty} a_n = a$.

Suppose $a_n \searrow$,

Since $\{a_n\}$ is bounded & non-empty, $\inf a_n = a$ exists by g.l.b. property.

i.e. $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $a_n - a < \varepsilon$ $\forall n > N$.

(Since otherwise $a+\varepsilon$ would be the true g.l.b.)

Hence, $|a - a_n| < \varepsilon$ $\forall n > N$ $\Rightarrow \lim a_n = a$.

Complete Space & Completion Theory

Converge seq. $\Rightarrow$ Cauchy sequence.

But Cauchy sequence $\not\Rightarrow$ converging sequence.

EX: In $\mathbb{Q}$, 3, 3.1, 3.14, 3.141, ⋯ $\longrightarrow \pi \notin \mathbb{Q}$.

Def: X is a Complete Metric Space iff
Every Cauchy sequence converges in X.

Properties:
1. Compact spaces are complete.
2. Closed subspaces of compact spaces are complete.
3. $\mathbb{Q}$ is not complete.
4. $\mathbb{R}^n$ is complete.

Completion Theory

Every metric space can be completed.

If M can be completed to $\hat{M}$, then M is a dense metric subspace of $\hat{M}$.

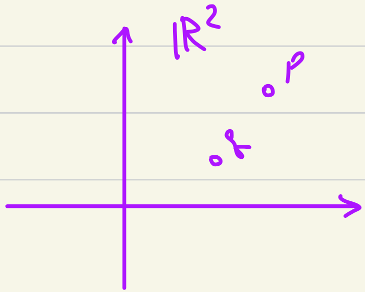
Example: $(0,1) \to [0,1]$

$\mathbb{R} \setminus \{0\} \to \mathbb{R}$

$\mathbb{Q} \longrightarrow \mathbb{R}$: Add $\{q_n : q_n^2 < 2\} \to \sqrt{2}$, etc.

$\mathbb{R}^2 \setminus \{p, q\} \longrightarrow \mathbb{R}^2$: Add $\{a_n \to p\} \longrightarrow p$

$\{b_n \to q\} \longrightarrow q$



Equivalence

Let $\mathcal{C}$ be the collection of Cauchy sequences in M.

$\forall \{a_n\}, \{b_n\}$, $d(\{a_n\}, \{b_n\}) = \lim_{n\to\infty} |a_n - b_n|$

$\Rightarrow \{a_n\} \sim \{b_n\}$ if $d(\{a_n\}, \{b_n\}) = 0$.

(equivalent)

Let $\hat{M} = C/\sim$ (call $p\sim q$ are the same point).

1. d is a well-defined metric on $\hat{M}$. Note that we change d to $D([\{p_n\}], [\{q_n\}]) = \lim_{n\to\infty} |p_n - q_n|$
2. $M \subseteq \hat{M}$
3. $\hat{M}$ is complete. (For proof, see Pugh, p. 119–121)
4. Uniqueness

In Rudin, it is exercise 24 (a) ~ (e)...

Limit - Sup

$$\limsup_{n\to\infty} a_n = \sup\{\text{subsequence limits of } a_n\} = \lim_{n\to\infty} \sup_{k\geq n} a_k$$

Example: $\{b_n\} = \{1, -1, \frac{1}{2}, -1, \frac{1}{3}, -1, \frac{1}{4}, \cdots\}$, $1, \frac{1}{2}, \frac{1}{3}, \cdots \to 0$

$$\limsup_{n\to\infty} b_n = 0, \quad \liminf_{n\to\infty} b_n = -1.$$

1. $\limsup a_n + \limsup b_n \neq \limsup (a_n + b_n)$

Counter-example:

$$\begin{array}{rlr}
 & 1, -1, 1, -1, \cdots & \limsup_{n\to\infty} = 1 \\
+) & -1, 1, -1, 1, \cdots & \limsup_{n\to\infty} = 1 \\
\hline
 & 0, 0, 0, 0, \cdots & \limsup_{n\to\infty} = 0
\end{array}$$

2. $c \cdot \left[\limsup_{n\to\infty} a_n\right] = \limsup_{n\to\infty} (c \cdot a_n)$ if $c > 0$

But if $c = -1$, $c \cdot \{1, -1, 1, -1, \cdots\}$ becomes $\{-1, 1, -1, 1, \cdots\}$, but both have $\limsup_{n\to\infty} = 1$

3. $\lim_{n\to\infty} a_n$ exists iff $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n$.