Online Math Camp (2325)

TA Session Note (4/17)

Heine-Borel Theorem

In $(\mathbb{R}^n, d)$, d: Euclidean metric

$$K \subseteq \mathbb{R}^n \text{ is compact} \iff K \text{ is bounded \& closed.}$$

(pf) $(\Rightarrow)$

$(\Leftarrow)$

Prop. K is compact $\iff$ $\forall$ infinite subset S of K, S has a limit point in K.

(pf) ($\Rightarrow$) S is an infinite subset that has no limit point.

Then $\forall p \in K$, $\exists U_p$: open set containing p such that U_p contains at most one element in S.

Clearly, $\bigcup_{p \in K} U_p \supseteq K$ is an open cover.

$\Rightarrow \exists$ finite subcover $\bigcup_{p' \in K} U_{p'}$ where each U_p contains at most one element in S

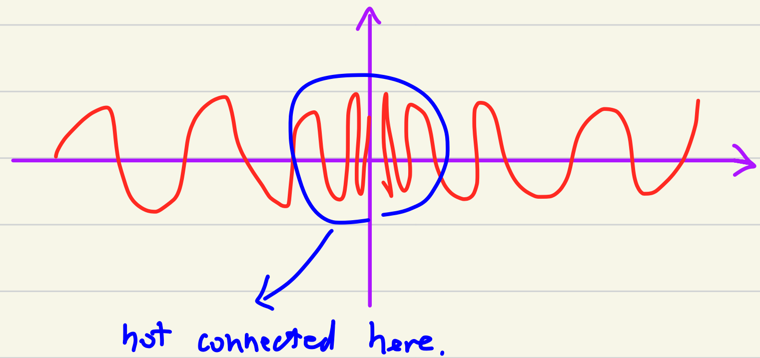
Hence, $\bigcup_{p' \in K} U_{p'}$ contains finite point in S ($\rightarrow\leftarrow$)

($\Leftarrow$) See video.

Connectedness

Def: C is connected iff $\nexists$ non-empty $A, B \subseteq C$ such that $A \cap \bar{B} = \phi$ and $B \cap \bar{A} = \phi$.

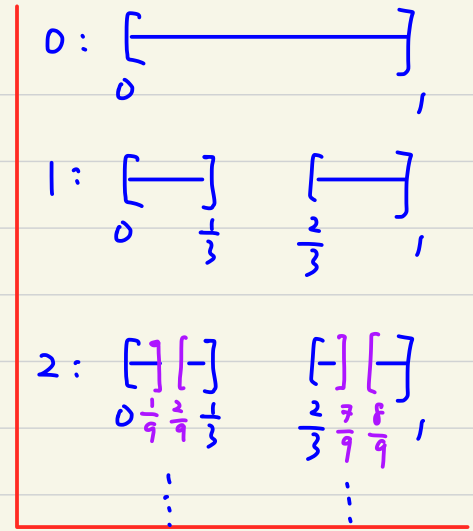
Example: $y = \sin\left(\frac{1}{x}\right)$



Cantor Set

三進位法

$$C = \{ 0.a_1 a_2 \cdots \mid \forall i,\ a_i = 0 \text{ or } 2 \}$$



Prop. 1: C is uncountable.

Prop. 2: C is closed.

Prop 3: C contains no isolate point.
i.e. Every point in C is a limit point.

(pf) For $0.a_1 a_2 \cdots \in C$, $\forall r > 0$, $\exists\, m \in \mathbb{N}$ such that $\frac{1}{3^m} < r$.

If we can find a point in $D_{\frac{1}{3^m}}(0.a_1 a_2 \cdots a_n \cdots) \setminus \{0.a_1 a_2 \cdots\}$, then we are done.

If $\begin{cases} a_{m+1} = 0, \\ a_{m+1} = 2 \end{cases}$ consider $\begin{cases} 0.a_1 a_2 \cdots a_m (2000\cdots) \Rightarrow d(0.a_1 \cdots a_m(200\cdots),\ 0.a_1 \cdots a_m(\underset{=a_{m+1}}{0}\cdots)) \leq 2 \cdot \frac{1}{3^{m+1}} \\ 0.a_1 a_2 \cdots a_m (2222\cdots) \Rightarrow d(0.a_1 \cdots a_m(222\cdots),\ 0.a_1 \cdots a_m(\underset{=a_{m+1}}{2}\cdots)) \leq 2 \cdot \frac{1}{3^{m+1}} \end{cases}$

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Example: What is $A = \{ a+b \mid a, b \in C = \text{Cantor set} \}$? $\Rightarrow [0, 2]$!!