Online Math Camp (23S)

TA Session Note (4/17)

(Quiz 7 Solution)

1. (25 pts) Define "C is a *connected set* in the metric space X".

Def: A set C is connected if C is not an union of two non-empty, separated set.

Or, $\nexists$ non-empty $A, B \in C$ such that $A \cap \bar{B} = \phi$, $B \cap \bar{A} = \phi$.

2. (1) (25 pts) State *Heine-Borel theorem.*

(2) (20 pts) Is $([a, b], d)$ compact where d denotes the discrete metric? Why you cannot use Heine-Borel in this case?

(1) In $\mathbb{R}^n$, K is compact iff K is closed & bounded.

(2) $([a,b], d)$ is not compact.

Consider $\{N_1(x) \mid \forall x \in [a,b]\}$.

Since $N_1(x) = \{x\}$, it has no finite subcover.

We cannot use Heine-Borel since $([a,b], d)$ is not in $\mathbb{R}^n$!

3. (24 pts) Prove that if a set is compact, then every infinite subset has a limit point.

H: compact. $S \subseteq H$ is an infinite subset.

W.T.S. S has a limit point.

(pf) If not, S has no limit point.

$\Rightarrow$ For $x \in S$, $\exists N_{\varepsilon_x}(x)$ such that $N_{\varepsilon_x}(x) \cap S = \{x\}$

Consider $\{S^c, \{N_{\varepsilon_x}(x) \mid \forall x \in S\}\}$ of H,

which has no finite subcover $\Rightarrow H$ is not compact ($\rightarrow\leftarrow$)

4. (24 pts) Show that the Cantor set is perfect, that is, closed and with no isolated point.

(i) Cantor set F is closed.

(pf) $F = \bigcap_{i=1}^{\infty} F_i$ is closed since F_i is closed $\forall i$.

(ii) Cantor set F has no isolated point.

(pf) Take $x \in F$, consider $(x-\varepsilon, x+\varepsilon)$, $\exists N \in \mathbb{N}$ such that $\frac{1}{3^N} < \varepsilon$

Define $M = \max\{m \mid \frac{m}{3^N} < x\} \Rightarrow [\frac{M}{3^N}, \frac{M+1}{3^N}] \subseteq (x-\varepsilon, x+\varepsilon)$

(i) $x \in [\frac{M}{3^N}, \frac{3M+1}{3^{N+1}}]$

(ii) $x \in [\frac{3M+2}{3^{N+1}}, \frac{M+1}{3^N}]$

implies that $\exists\, c \neq x$ such that $c \in F$ & $c \in N_\varepsilon(x)$

#

closure of C

5. (20 pts) Prove that, if C is connected, then $\overline{C}$ is also connected. How about the inverse?

(pf) $\overline{C} = A \cup B$, A and B are separated. <u>Claim</u>: Either A or B is empty.

We know $C \subseteq \overline{C} = A \cup B$. $\Rightarrow C = (A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

Since $A \cap C$ & $B \cap C$ are separated and C is connected, either $A \cap C$ or $B \cap C$ is empty.

If $A \cap C = \phi$ $\Rightarrow C = B \cap C$ since $C = \cancel{(A \cap C)}^{\phi} \cup (B \cap C)$

$\Rightarrow C \subseteq B \Rightarrow \overline{C} \subseteq \overline{B}$

Then $A = A \cap (A \cup B) = A \cap \overline{C} \subseteq A \cap \overline{B}$ (since $\overline{C} \subseteq \overline{B}$)

$= \phi$ (since A & B are separated).

(Inverse) Let $C = [0, 1] \setminus \{\frac{1}{2}\}$,

$\overline{C}$ is connected, but C is not. $(\!\longrightarrow\!)(\!\longleftarrow\!)$