Online Math Camp (23S)

TA Session Note (4/10)

Compactness

Def: K is compact iff

$\forall$ open cover of K, $\bigcup_{\alpha \in A} U_\alpha \; (\supseteq K)$

$\exists$ finite subcover $\bigcup_{\alpha' \in A'} U_{\alpha'}$, $A' \subseteq A$ is finite, such that $\bigcup_{\alpha' \in A'} U_{\alpha'} \supseteq K$.

Intuition: When f is continuous on compact set K

$\Downarrow$

$\forall \varepsilon > 0, \exists \delta > 0$ such that $|x - x'| < \delta \Rightarrow |f(x) - f(x')| < \varepsilon$.

Set $\varepsilon > 0$, for each point $x_0 \in K$, can find $\delta(x_0)$ such that $|f(x) - f(x_0)| < \varepsilon \Rightarrow$ These open balls $(x_0 - \delta(x_0), x_0 + \delta(x_0))$ covers K.

$\Rightarrow$ Need only finite subset of this to cover K $\Rightarrow |f(x)| < f(x_0) + 2\bar{N}\varepsilon$.

say, $\bar{N}$ of them

$\Rightarrow f$ is bounded.

Examples:

1. $(0,1)$ is not compact: $\left(\frac{1}{n}, 1-\frac{1}{n}\right) \subseteq (0,1) \quad \forall n \geq 3$,

$$\Rightarrow \bigcup_{n \in \mathbb{N} \setminus \{1,2\}} \left(\frac{1}{n}, 1-\frac{1}{n}\right) \supseteq (0,1),$$ but no finite subset covers $(0,1)$.

2. empty set is compact since any cover would contain it!

Proposition: S is compact $\Rightarrow$ S is bounded.

ambient space (like $\mathbb{R}^n$)

(pf) Suppose S is not bounded in (X, d).

Pick $p \in X$ and open ball $N_n(p)$ for $n \in \mathbb{N}$ such that $\bigcup_{n \in \mathbb{N}} N_n(p) \supseteq S$.

For any finite subcover $\bigcup_{n' \in A} N_{n'}(p)$, there exists a maximal in A, a, such that $\bigcup_{n' \in A} N_{n'}(p) = N_a(p) \not\supseteq S$ since S is not bounded.

Prop. S is compact $\Rightarrow$ S is closed.

(pf) Suppose S is not closed, then $\exists\, q \notin S$ as a limit point of S.

Consider open cover $\bigcup_{p\in S} N_{\frac{d(p,q)}{2}}(p) \supseteq S$.

For all finite cover, $K = \bigcup_{p'\in S'} N_{\frac{d(p',q)}{2}}(p')$, pick $r \in K$,

$d(r,q) \geq d(p',q) - d(r,p')$ (triangular inequality)

$> d(p',q) - \frac{d(p',q)}{2} = \frac{d(p',q)}{2} \geq \min_{p'\in S'} \frac{d(p',q)}{2} =: \bar{k}$

But since q is a limit point of S,

there exists $r' \in S$ such that $d(r',q) < \bar{k}$. $\Rightarrow r' \notin K$. $(\rightarrow\!\leftarrow)$.

Prop. For $K \subseteq Y \subseteq X$, $\quad K$ is compact in $X \Rightarrow K$ is compact in Y.

$$K \underset{\text{cpt}}{\subseteq} (X, d) \qquad\qquad K \underset{\text{cpt}}{\subseteq} (Y, d)$$

(pf) For any open cover $\bigcup_{\alpha \in A} U_\alpha \subseteq K$, $\quad U_\alpha \in Y \subseteq X$

$\exists$ finite subcover $\bigcup_{\alpha' \in A'} U_{\alpha'} \subseteq K$ since K is compact in X.

Hence, K is compact in Y. #

Prop. Any closed subset of a compact set is closed.

(pf) Consider $F \subseteq K$, F: closed, K: compact.

open since F is closed.

For any open cover $\bigcup_{\alpha \in A} U_\alpha$ of F, $\left(\bigcup_{\alpha \in A} U_\alpha\right) \cup F^c$ is an open cover of K.

Since K is compact, $\exists$ finite subcover $\left(\bigcup_{\alpha \in A'} U_{\alpha'}\right) \cup F^c \; (\supseteq K)$

$\Rightarrow \bigcup_{\alpha \in A'} U_{\alpha'} \; (\supseteq F)$ is a finite subcover of F. #