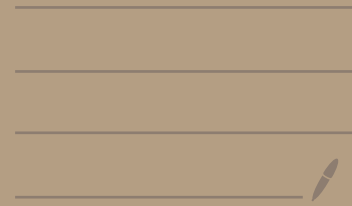


Online Math Camp (235)

TA Session Note (4/10)



Compactness

Def: K is compact iff

\forall open cover of K , $\bigcup_{\alpha \in A} U_{\alpha} \supseteq K$

\exists finite subcover $\bigcup_{\alpha' \in A'} U_{\alpha'}$, $A' \subseteq A$ is finite, such that $\bigcup_{\alpha' \in A'} U_{\alpha'} \supseteq K$.

Intuition: When f is continuous on compact set K



$\forall \varepsilon > 0, \exists \delta > 0$ such that $|x - x'| < \delta \Rightarrow |f(x) - f(x')| < \varepsilon$.

Set $\varepsilon > 0$, for each point $x_0 \in K$, can find $\delta(x_0)$ such that $|f(x) - f(x_0)| < \varepsilon \Rightarrow$ These open balls $(x_0 - \delta(x_0), x_0 + \delta(x_0))$ covers K .

\Rightarrow Need only finite subset of this to cover $K \Rightarrow |f(x)| < f(x_0) + 2\bar{N}\varepsilon$.
say, \bar{N} of them $\Rightarrow f$ is bounded.

Examples:

1. $(0, 1)$ is not compact: $(\frac{1}{n}, 1 - \frac{1}{n}) \subseteq (0, 1) \quad \forall n \geq 3,$

$\Rightarrow \bigcup_{n \in \mathbb{N} \setminus \{1, 2\}} (\frac{1}{n}, 1 - \frac{1}{n}) \supseteq (0, 1),$ but no finite subset covers $(0, 1)$.

2. empty set is compact since any cover would contain it!

Proposition: S is compact $\Rightarrow S$ is bounded.

(pf) Suppose S is not bounded in (X, d) .

ambient space (like \mathbb{R}^n)

Pick $p \in X$ and open ball $N_n(p)$ for $n \in \mathbb{N}$ such that $\bigcup_{n \in \mathbb{N}} N_n(p) \supseteq S$.

For any finite subcover $\bigcup_{n' \in A} N_{n'}(p)$, there exists a maximal in A , a ,

such that $\bigcup_{n' \in A} N_{n'}(p) = N_a(p) \not\supseteq S$ since S is not bounded.

Prop. S is compact $\Rightarrow S$ is closed.

(pf) Suppose S is not closed, then $\exists q \notin S$ as a limit point of S .

Consider open cover $\bigcup_{p \in S} N_{\frac{d(p, q)}{2}}(p) \supseteq S$.

For all finite cover, $K = \bigcup_{p' \in S'} N_{\frac{d(p', q)}{2}}(p')$, pick $r \in K$,

$$d(r, q) \geq d(p', q) - d(r, p') \quad (\text{triangular inequality})$$

$$> d(p', q) - \frac{d(p', q)}{2} = \frac{d(p', q)}{2} \geq \min_{p' \in S'} \frac{d(p', q)}{2} = \bar{k}$$

But since q is a limit point of S ,

there exists $r' \in S$ such that $d(r', q) < \bar{k} \Rightarrow r' \notin K$. ($\rightarrow \times$)

Prop. For $K \subseteq Y \subseteq X$, K is compact in $X \Rightarrow K$ is compact in Y .

$K \underset{\text{cpt}}{\subseteq} (X, d)$

$K \underset{\text{cpt}}{\subseteq} (Y, d)$

(pf) For any open cover $\bigcup_{\alpha \in A} U_{\alpha} \subseteq K$, $U_{\alpha} \in Y \subseteq X$

\exists finite subcover $\bigcup_{\alpha' \in A'} U_{\alpha'} \subseteq K$ since K is compact in X .

Hence, K is compact in Y . $\#$

Prop. Any closed subset of a compact set is closed.

(Pf) Consider $F \subseteq K$, F : closed, K : compact.

For any open cover $\bigcup_{\alpha \in A} U_\alpha$ of F , $\left(\bigcup_{\alpha \in A} U_\alpha\right) \cup \underline{F^c}$ is an open cover of K .
 open since F is closed.

Since K is compact, \exists finite subcover $\left(\bigcup_{\alpha \in A'} U_{\alpha'}\right) \cup F^c \left(\supseteq K\right)$

$\Rightarrow \bigcup_{\alpha \in A'} U_{\alpha'} \left(\supseteq F\right)$ is a finite subcover of F . $\#$