Online Math Camp (2025)

TA Session Notes (3/27)

Open Sets

Intuition: "something near"

In Calculus, we define $\lim_{x \to a} f(x) = L$ as:

$\forall \varepsilon > 0$, $\exists \delta > 0$, such that $\forall\, 0 < |x' - x| < \delta$, $\Rightarrow |f(x') - f(x)| < \varepsilon$

$x' \in N_\delta(x) \Rightarrow f(x') \in N_\varepsilon(f(x))$

Def: A set U is open

r-neighborhood

iff $\forall x \in U$, $\exists r > 0$, such that $N_r(x) = \{x' \mid d(x', x) < r\} \subseteq U$.

Closed Sets

Intuition: $\{a_n\} \subseteq F$, closed, converges. $\Rightarrow \lim a_n \in F$.

Def: F is closed iff all limit points of F lies in F.

p is a limit point iff $\forall\, \delta > 0$, $\exists\, x \in F$ such that $x \in N_\delta(p)$.

Examples of Open & Closed Sets

1. $N_r(x) = \{x' \mid d(x', x) < r\}$ is open

(pf) For all $x' \in N_r(x)$, take $r' = r - d(x, x')$. We want to show:

$$N_{r'}(x') \subseteq N_r(x) \iff \forall x'' \text{ with } \underbrace{d(x'', x') < r'}_{\Updownarrow \atop x'' \in N_{r'}(x')} \Rightarrow \underbrace{d(x'', x) < r}_{\Updownarrow \atop x'' \in N_r(x)}$$

Since $d(x'', x) \leq \overbrace{d(x'', x')}^{< r'} + d(x', x) < r' + d(x', x) = r$, we are done. #

2. ϕ is both open & closed.

(pf) Suppose ϕ is not open. Then, $\exists\, x \in \phi$ such that $\forall r>0$, $N_r(x) \not\subseteq \phi$.

But since ϕ is empty, no such x exists. Thus, ϕ is open.

Suppose ϕ is not closed. Then, $\exists \{a_n\} \in \phi$ such that $\lim a_n = a \notin \phi$.

But since ϕ is empty, no such $\{a_n\}$ exists. Thus, ϕ is closed. //

Proposition 1. Finite intersections of open sets are open.

(pf) U_i is open for $1 \le i \le n$. $\Longleftrightarrow$ $\forall x \in U_i$, $\exists\, r_{i,x} > 0$ such that $N_{r_{i,x}}(x) \subseteq U_i$

We want to show $\bigcap_{i=1}^{n} U_i$ is open

$$\Longleftrightarrow \forall x \in \bigcap_{i=1}^{n} U_i,\ \exists\, r>0,\ \text{s.t. } N_r(x) \subseteq \bigcap_{i=1}^{n} U_i$$

Pick $r = \min_i r_{i,x}$, then $N_r(x) \subseteq N_{r_{i,x}}(x) \subseteq U_i \quad \forall i$

$$\Rightarrow N_r(x) \subseteq \bigcap_{i=1}^{n} U_i$$ #

Remark: Some intersections of open sets are not open:

For example, $\bigcap_{n \in \mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$ is closed,

(Others are neither open, nor closed)

Proposition 2. Any union of open sets is open.

Proposition U is open $\iff$ U^c is closed.