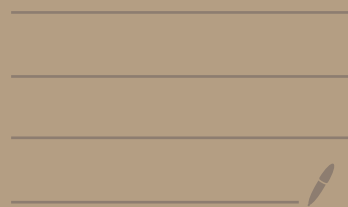


Online Math Camp (235)

TA Session Notes (3/27)



Open Sets

Intuition: "something near"

In Calculus, we define $\lim_{x \rightarrow a} f(x) = L$ as:

$$\forall \epsilon > 0, \exists \delta > 0, \text{ such that } \forall 0 < |x' - x| < \delta, \Rightarrow |f(x') - f(x)| < \epsilon$$
$$x' \in N_\delta(x) \Rightarrow f(x') \in N_\epsilon(f(x))$$

Def: A set U is open

iff $\forall x \in U, \exists r > 0$, such that $N_r(x) = \{x' \mid d(x', x) < r\} \subseteq U$.

r-neighborhood

Closed Sets

Intuition: $\{a_n\} \subseteq F$, closed, converges. $\Rightarrow \lim a_n \in F$.

Def: F is closed iff all limit points of F lies in F .

\downarrow
 p is a limit point iff $\forall \delta > 0, \exists x \in F$ such that $x \in \mathcal{N}_\delta(p)$.

Examples of Open & Closed Sets

1. $N_r(x) = \{x' \mid d(x', x) < r\}$ is open

(pf) For all $x' \in N_r(x)$, take $r' = r - d(x, x')$. We want to show:

$$N_{r'}(x') \subseteq N_r(x) \iff \forall x'' \text{ with } \underbrace{d(x'', x') < r'}_{\substack{\Downarrow \\ x'' \in N_{r'}(x')}} \Rightarrow \underbrace{d(x'', x) < r}_{\substack{\Downarrow \\ x'' \in N_r(x)}}$$

Since $d(x'', x) \leq \overset{< r'}{d(x'', x')} + d(x', x) < r' + d(x', x) = r$, we are done. \ast

2. \emptyset is both open & closed.

(pf) Suppose \emptyset is not open. Then, $\exists x \in \emptyset$ such that $\forall r > 0, N_r(x) \not\subseteq \emptyset$.

But since \emptyset is empty, no such x exists. Thus, \emptyset is open.

Suppose \emptyset is not closed. Then, $\exists \{a_n\} \in \emptyset$ such that $\lim a_n = a \notin \emptyset$.

But since \emptyset is empty, no such $\{a_n\}$ exists. Thus, \emptyset is closed. ~~#~~

Proposition 1. Finite intersections of open sets are open.

(pt) U_i is open for $1 \leq i \leq n$. $\Leftrightarrow \forall x \in U_i, \exists r_{i,x} > 0$ such that $N_{r_{i,x}}(x) \subseteq U_i$

We want to show $\bigcap_{i=1}^n U_i$ is open

$$\Leftrightarrow \forall x \in \bigcap_{i=1}^n U_i, \exists r > 0, \text{ s.t. } N_r(x) \subseteq \bigcap_{i=1}^n U_i$$

Pick $r = \min_i r_{i,x}$, then $N_r(x) \subseteq N_{r_{i,x}}(x) \subseteq U_i \quad \forall i$

$$\Rightarrow N_r(x) \subseteq \bigcap_{i=1}^n U_i \quad \#$$

Remark: Some intersections of open sets are not open:

For example, $\bigcap_{n \in \mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$ is closed.

(Others are neither open, nor closed)

Proposition 2. Any union of open sets is open.

Proposition U is open $\Leftrightarrow U^c$ is closed.