Online Math Camp (2023)

TA Session Note (3/27)

(Quiz 5 Solution)

1. (15 pts each) Give formal definitions to the following statements.

   (i) U is an *open set* in a metric space X.

   (ii) F is an *closed set* in a metric space X.

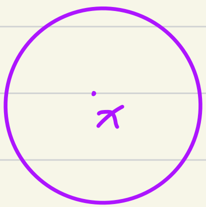
(See review note)

2. (12 pts each) Is the set S open in X? Is it closed? Explanations are needed.

(i) $X = \mathbb{R}^2$. S is some open ball $N_r(x)$ for $r > 0$.

S is open by triangular inequality.

S is not closed. since the circle are limit points of S, but not in S.



(ii) S is X itself.

$\forall p \in S, \quad \forall N(p) \subset X = S \Rightarrow S$ is open.

$\forall$ limit point of S are in $X = S \Rightarrow S$ is closed.

(iii) S is an empty set.

See review notes.

(iv) $X = \mathbb{R}^5$. S is a non-empty finite set.

$S = \{x_1, \cdots, x_n\}$. let $r = \min\{|x_i - x_j| \mid i \neq j\}$.

For x, $N_r(x) \nsubseteq S$. $\Rightarrow \nexists N_q(x) \subseteq S$. $\Rightarrow$ S is not open.

Since S is finite, S has no limit point $\Rightarrow$ S is not closed.

3. (30 pts) Prove that S is open in X if and only if S^c is closed in X.

($\Rightarrow$) If S is open. $\forall$ limit point p in S^c,

$\forall N_r(p)$ $\exists$ $q \in N_r(p)$ such that $q \neq p$, $q \in S^c$.

$\Rightarrow$ p is not an interior point of S.

(Otherwise, $\exists N(p)$ such that $N(p) \subseteq S \Rightarrow N(p) \cap S^c = \phi$)

$\Rightarrow p \notin S \Rightarrow p \in S^c$, i.e. S^c is closed.

($\Leftarrow$) If S^c is closed, $\forall p \in S$, $\Rightarrow p \notin S^c$.

i.e. p is not a limit point of S^c.

$\exists$ $N(p)$ such that $N(p) \cap S^c = \phi$.

$\Rightarrow N(p) \subseteq S$.

i.e. S is open $\#$

4. (30 pts) Show that the union of any collection of open sets is open.

$\{U_i\}_{i \in I}$: U_i is open $\forall i$. claim $\bigcup_{i \in I} U_i$ is open.

(pf) $\forall p \in \bigcup_{i \in I} U_i$, $\exists t$ such that $p \in U_t$. (by definition of union).

Since U_t is open, $\exists N(p)$ such that $N(p) \subseteq U_t \subseteq \bigcup_{i \in I} U_i$.

Hence, $\bigcup_{i \in I} U_i$ is open. #

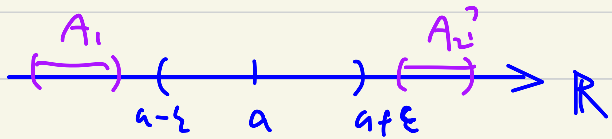
5. (**0 pts, don't do this unless you have time**) Prove that a bounded closed set of real numbers contains its supremum and infimum.

Bounded Closed set $A \subseteq \mathbb{R}$. Claim: $\sup A = a \in A$, $\inf A = b \in A$.

(pf) Suppose $a \notin A$. (show that a is a limit point to get contradiction)

If a is not a limit point of A, then $\exists\ (a-\varepsilon, a+\varepsilon) \cap A = \phi$.

$\Rightarrow$ Either "$a-\varepsilon$ is an upper bound of A" or "a is an upper bound of A."



If $A_2 \subseteq A$, then a is not upper bound

If $A_2 \notin A$, then any point in $(a-\varepsilon, a+\varepsilon)$ is an upper bound.

$\Rightarrow$ Either statement contradicts $a = \sup A$. $(\rightarrow\leftarrow)$