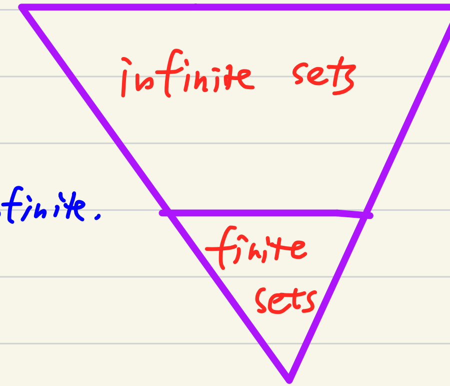
Online Math Camp (235)

TA Session Notes (3/20)

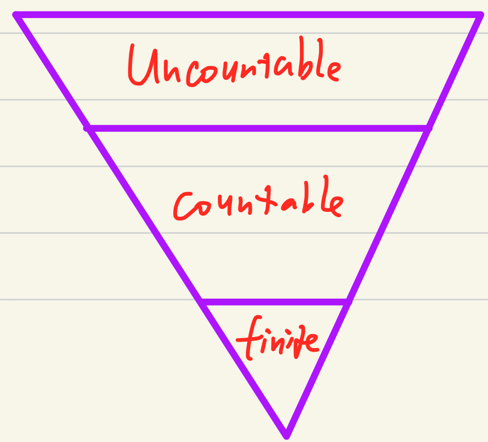
Cardinality

How big is a set? (in magnitude)

If A is infinite, B is finite, then $A \setminus B$ is infinite.



If A is uncountable, $B \subseteq A$ is at most countable, then $A \setminus B$ is uncountable.



Proposition 1: (Countable Sets are the Smallest infinite sets)

A: infinite $\Rightarrow \exists A' \subsetneq A$, A': countable.

(pf) Pick an element $a_1 \in A$

Pick an element $a_2 \in A \setminus \{a_1\}$

$\vdots$

Since A is infinite, this procedure can go infinitely.

Hence, the set $\{a_1, a_2, \cdots, a_n, \cdots\}$ is countable.

Proposition 2 (Infinite subsets of countable sets are countable)

The subsets of countable sets is at most countable.

(pf) Since A is countable, $f: A \longrightarrow \mathbb{N}$ is a bijection.

(sketch) i.e. $A = \{a_1, a_2, \cdots, a_n, \cdots\cdots\}$

Hence, $B \subseteq A \Rightarrow B = \{a_{n_1}, a_{n_2}, \cdots\}$ is countable (bijection).

Proposition 3 Countable union of countable sets is countable.

(pf) $A_1 = \{a_{11}, a_{12}, \cdots, a_{1n}, \cdots\}$

$A_2 = \{a_{21}, a_{22}, \cdots, a_{2n}, \cdots\}$

$A_3 = \{a_{31}, a_{32}, \cdots, a_{33}, \cdots\}$

$\vdots$

$\Rightarrow \bigcup_{i=1}^{\infty} A_i = \{a_{11}, a_{12}, a_{21}, a_{31}, a_{22}, a_{13}, \cdots\}$ countable.

Example: ① $\bigcup_{n\in\mathbb{N}} \underbrace{\{(m,n) \mid m\in\mathbb{Z}\}}_{\text{countable}} = \underbrace{\{(m,n) \mid m\in\mathbb{Z}, n\in\mathbb{N}\}}_{\text{countable}}$

② $\bigcup_{n\in\mathbb{N}} \underbrace{[0, 1-\tfrac{1}{n})}_{\text{uncountable}} = \underbrace{[0, 1)}_{\text{uncountable}}$

Cor. $\mathbb{Q}$ is countable.

(pf) Since $\mathbb{Q}$ can be viewed as $\{(m,n) \mid m\in\mathbb{Z}, \underbrace{g.c.d.}_{\text{greatest common denominator}}(m,n)\in\{1,0\}\}$

$\mathbb{Q}$ is countable like the example above.

Proposition 4: $(0,1)$ is uncountable.

(pf) If not, $f : \mathbb{N} \longrightarrow (0,1)$ is a bijection.

Then,
$$1 \longmapsto 0.a_{11}\, a_{12}\, a_{13} \cdots$$
$$2 \longmapsto 0.a_{21}\, a_{22}\, a_{23} \cdots$$
$$3 \longmapsto 0.a_{31}\, a_{32}\, a_{33} \cdots$$
$$\vdots$$

Consider the number $0.b_1 b_2 b_3 \cdots$ where $b_i = \begin{cases} 1 & \text{if } a_{ii} = 0 \\ 0 & \text{if } a_{ii} \neq 0. \end{cases}$

Then, $0.b_1 b_2 b_3 \cdots$ is <u>NOT</u> in this bijection.

Corr. $\mathbb{R}$ is uncountable.

(pf) Since $(0,1)$ is uncountable, $\mathbb{R} \supseteq (0,1)$ cannot be countable.

Metric Space

Def: (X, d) is a metric space if ① $d(x,y) \geq 0$ ("=" iff $x=y$)

② $d(x,y) = d(y,x)$

③ $d(x,y) + d(y,z) \geq d(x,z)$

Example 1: $(\mathbb{R}^2, d)$, d: Euclidean

(pf) ①, ② are trivial, ③ requires

$$\sqrt{(x_1-y_1)^2+(x_2-y_2)^2} + \sqrt{(y_1-z_1)^2+(y_2-z_2)^2} \geq \sqrt{(x_1-z_1)^2+(x_2-z_2)^2}$$

$$\Leftrightarrow \underbrace{(x_1-y_1)}_{a_1}{}^2 + \underbrace{(x_2-y_2)}_{a_2}{}^2 + \underbrace{(y_1-z_1)}_{b_1}{}^2 + \underbrace{(y_2-z_2)}_{b_2}{}^2 + \underbrace{2\sqrt{\cdot}\,\sqrt{\cdot}}_{2\sqrt{(a_1^2+a_2^2)(b_1^2+b_2^2)}} \geq \underbrace{(x_1-z_1)^2}_{(a_1+b_1)^2} + \underbrace{(x_2-z_2)^2}_{=(a_2+b_2)^2}$$

$$\Rightarrow a_1^2+a_2^2+b_1^2+b_2^2+2\sqrt{(a_1^2+a_2^2)(b_1^2+b_2^2)} \geq a_1^2+a_2^2+b_1^2+b_2^2+2(a_1b_1+a_2b_2)$$

$$= (a_1+b_1)^2+(a_2+b_2)^2 \quad \#$$