Online Math Camp (23年)

TA Session Notes (3/20)

(Quiz 4 Solution)

1. (25 pts) Give formal definitions to the statement "(X, d) *is a metric space*".

2. (25 pts) Let $X = [0, \infty)$. Is $d(x, y) = \left(\sqrt{x} - \sqrt{y}\right)^2$ a metric on X? Prove or disprove.

(Sol) For $x = 9$, $y = 1$, $z = 4$, $d(x, z) + d(z, y) = 1 + 1 < 4 = d(x, y)$

$(\rightarrow\leftarrow)$

(Sol 2) $d(x, z) + d(z, y) - d(x, y)$

$$= (\sqrt{x} - \sqrt{z})^2 + (\sqrt{z} - \sqrt{y})^2 - (\sqrt{x} - \sqrt{y})^2$$

$$= \cancel{x} + z - 2\sqrt{xz} + z + \cancel{y} - 2\sqrt{yz} - \cancel{x} - \cancel{y} + 2\sqrt{xy}$$

$$= 2z - 2\sqrt{xz} - 2\sqrt{yz} + 2\sqrt{xy} = 2(\sqrt{z} - \sqrt{x})(\sqrt{z} - \sqrt{y})$$

> 0 iff $z > x, y$ or $z < x, y$ $(\rightarrow\leftarrow)$

3. (15 pts each) Countable or Uncountable? Explain it in a few lines. (No need to be too rigorous, just to make sure you are not guessing.)

(i) The set of irrational numbers.

(ii) The set of infinite sequences with terms = 0 or 1.

(i) $\mathbb{R} \setminus \mathbb{Q}$: Since $\mathbb{R}$ is uncountable, $\mathbb{Q}$ is countable,

$\Rightarrow \mathbb{R} \setminus \mathbb{Q}$ is countable.

(ii) $S = \{x \mid x = (x_1, x_2, \cdots), x_n \in \{0, 1\}, \forall n \in \mathbb{N}\}$

Suppose S is countable, then there exists bijection $f: \mathbb{N} \longrightarrow S$

Let $y = (y_1, y_2, \cdots, y_n, \cdots)$, $y_n \in \{0, 1\}$, $y_n \neq x_n^n$ $\forall n \in \mathbb{N}$.

$\left\{\begin{array}{l} f(1) = (x_1^1, x_2^1, \cdots x_n^1, \cdots) \\ f(2) = (x_1^2, x_2^2, \cdots, x_n^2, \cdots) \\ \vdots \end{array}\right.$

$\Rightarrow y \in S$, but there is no $n \in \mathbb{N}$ such that $f(n) = y$

by our construction of $y_n \neq x_n^n$. $(\rightarrow\leftarrow)$

4. (30 pts) Prove that $(\mathbb{R}^n, d)$ is a metric space for $n \in \mathbb{N}$, where d denotes the Euclidean metric.

(Sol) $d(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$.

(i) $\forall x, y \in R^n$, $d(x,y) \geq 0$, $d(x,y) = 0$ iff $x_i = y_i \ \forall i$

(ii) $d(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} = \sqrt{\sum_{i=1}^{n} (y_i - x_i)^2} = d(y,x)$.

(iii) $\underbrace{d(x,z) + d(z,y)}_{\parallel} \geq d(x,y) = RHS$ $\Rightarrow RHS^2 = \sum_i (x_i - y_i)^2$

$= \sum_i (x_i - z_i)^2 + \sum_i (z_i - y_i)^2$

$+ 2\underline{\sum (x_i - z_i)(z_i - y_i)}$

$LHS = \sqrt{\sum (x_i - z_i)^2} + \sqrt{\sum (z_i - y_i)^2}$

$\Rightarrow LHS^2 = \sum (x_i - z_i)^2 + \sum (z_i - y_i)^2 + \underline{2\sqrt{\cdot}\sqrt{\cdot}}$

$\geq$ is Cauchy-Swarz !!

$\geq \sum_i \left[(x_i - z_i) + (z_i - y_i)^2 \right] = \sum_i (x_i - y_i)^2 = RHS^2$ #

5. (28 pts) Prove that the set of all polynomials

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

with integral coefficients is countable. Deduce the set of algebraic numbers is countable. (An algebraic number is a number which is a root of a polynomial with integral coefficients.)

(Sol) $P = \{\text{polynomials with integral coefficients}\}$

$P_n = \{p \in P,\ \deg(p) \le n\} \Rightarrow P = \bigcup_{n \in \mathbb{N}} P_n$

<u>Claim:</u> $P_n \simeq \mathbb{Z}^{n+1}$

(pf) $f : P_n \longrightarrow \mathbb{Z}^{n+1}$ is a bijection $a_n x^n + \cdots a_1 x + a_0 \mapsto (a_n, a_{n-1}, \cdots a_0)$ #

Then, $P = \bigcup_{n \in \mathbb{N}} P_n \simeq \bigcup_{n \in \mathbb{N}} \mathbb{Z}^{n+1}$ is countable.

$\{\text{algebraic numbers}\} = \bigcup_{\underset{\text{countable}}{p \in \underline{P}}} \underbrace{\{\text{roots of } p\}}_{\text{at most countable.}}$ is countable. #

But, why $\underset{[0,1]}{\overset{\parallel}{A}} \not\subseteq \bigcup_{n\in\mathbb{N}} A_n$, $A_n = \{0.a_1a_2\cdots a_n,\ a_i \in \{0,1,2,\cdots,9\}\}$?

$\Rightarrow$ Because $\frac{1}{3} = 0.33\cdots\cdots \notin \bigcup_{n\in\mathbb{N}} A_n$ as all these have finite digits!!

Useful Tricks

<u>Proposition 1</u>: A is uncountable, B is countable.

$\Rightarrow |A \cup B| = |A|$. i.e. $\exists$ bijection $f : A \cup B \longrightarrow A$.

(pf) Take $C \subseteq A$, countable.

$\exists$ $f_1 : A \setminus C \longrightarrow A \setminus C$, bijection

$f_2 : B \cup C \longrightarrow B$, bijection

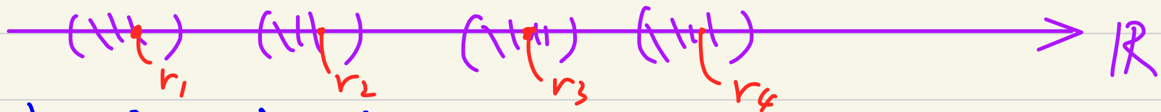
$\Rightarrow$ Combine f_1 & f_2 to form $f(x) = \begin{cases} f_1(x) & \text{if } x \in A \setminus C \\ f_2(x) & \text{if } x \in B \cup C. \end{cases}$

$\Rightarrow f$ is a bijection. #

<u>Homework</u>: $A \supseteq B$, A: uncountable, B: countable

Prove that $|A \setminus B| = |A|$.

Proposition 2 $\mathcal{S}$ contains disjoint open intervals on $\mathbb{R}$, then $\mathcal{S}$ is at most countable.



$$\mathcal{S} = (a_1, b_1) \cup (a_2, b_2) \cup \cdots \cup (a_n, b_n) \cup \cdots$$

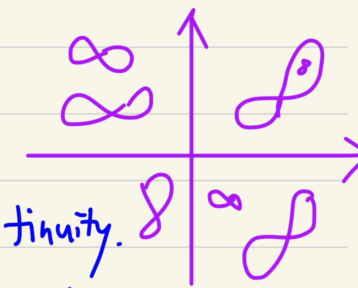
where $a_1 < b_1 < a_2 < b_2 < a_3 < b_3 < \cdots < a_n < b_n < \cdots$

(pf) Since every interval contains a rational number, we can send intervals $s \in \mathcal{S}$ to some $r \in \mathbb{Q} \cap \mathcal{S}$.

$\Rightarrow |\mathcal{S}| \le |\mathbb{Q}|$ #

Homeworks ① $f : \mathbb{R} \to \mathbb{R}$ is monotone.

Prove that there are at most countable discontinuity.



② Consider E, a set of "8" on $\mathbb{R}^n$ that do not overlap. Show that E is at most countable.