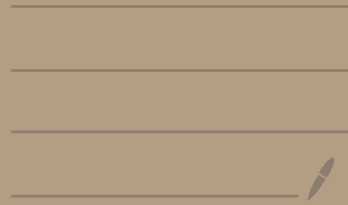


Online Math Camp (235)

TA Session Notes ($\frac{3}{13}$)



$$\inf E = \sup (-\bar{E})$$

Cauchy-Schwarz Inequality

$$\forall a, b \in \mathbb{C}^n, |a|^2 \cdot |b|^2 \geq |\langle a, b \rangle|^2$$

(pt) For \mathbb{R}^2 : Need to show $(a_1^2 + a_2^2)(b_1^2 + b_2^2) \geq (a_1 b_1 + a_2 b_2)^2$

$$\Leftrightarrow \underline{a_1^2 b_1^2} + a_2^2 b_1^2 + a_1^2 b_2^2 + \underline{a_2^2 b_2^2} \geq \underline{a_1^2 b_1^2} + \underline{2 a_1 b_1 a_2 b_2} + \underline{a_2^2 b_2^2}$$

$$\Leftrightarrow a_2^2 b_1^2 + a_1^2 b_2^2 - \underline{2 a_2 b_1 a_1 b_2} \geq 0$$

$$\Leftrightarrow (a_2 b_1 - a_1 b_2)^2 \geq 0 \quad \#$$

For \mathbb{R}^n : To show $(\sum a_i^2)(\sum b_i^2) \geq (\sum a_i b_i)^2$,

$$\text{Consider } \sum (a_i - b_i x)^2 = (\sum b_i^2) x^2 - 2(\sum a_i b_i) x + (\sum a_i^2) \geq 0$$

$$\text{Let } x = \frac{\langle a, b \rangle}{\langle b, b \rangle} \Rightarrow \langle b, b \rangle \cdot \frac{\langle a, b \rangle^2}{\langle b, b \rangle^2} - 2 \langle a, b \rangle \cdot \frac{\langle a, b \rangle}{\langle b, b \rangle} + \langle a, a \rangle \geq 0$$

$$\Rightarrow \langle a, a \rangle - \frac{\langle a, b \rangle^2}{\langle b, b \rangle} \geq 0 \Leftrightarrow |a|^2 \cdot |b|^2 \geq |\langle a, b \rangle|^2 \quad \#$$

(pf) (Continued) For \mathbb{C}^n ,

$$\text{consider } \sum \langle (a_i - b_i x), (a_i - b_i x) \rangle = \sum (a_i - b_i x) \overline{(a_i - b_i x)}$$

$$= \sum (a_i \bar{a}_i - \bar{a}_i b_i x - a_i \bar{b}_i \bar{x} + b_i \bar{b}_i x \bar{x})$$

$$= \sum [\langle a_i, a_i \rangle - \langle b_i, a_i \rangle x - \langle a_i, b_i \rangle \bar{x} + \langle b_i, b_i \rangle \langle x, x \rangle]$$

$$= \langle a, a \rangle - \langle b, a \rangle x - \langle a, b \rangle \bar{x} + \langle b, b \rangle \langle x, x \rangle \geq 0$$

$$\text{Let } x = \frac{\langle a, b \rangle}{\langle b, b \rangle} \Rightarrow \langle a, a \rangle - \frac{|\langle a, b \rangle|^2}{\langle b, b \rangle} - \frac{|\langle a, b \rangle|^2}{\langle b, b \rangle} + \langle b, b \rangle \frac{|\langle a, b \rangle|^2}{\langle b, b \rangle^2}$$

$$= \langle a, a \rangle - \frac{|\langle a, b \rangle|^2}{\langle b, b \rangle} \geq 0 \quad \#$$

Well Ordered Principle (WOP)

$\forall S \subseteq \mathbb{N}, S \neq \emptyset \Rightarrow S$ has the least element

Principle of Induction (PI)

If $S \subseteq \mathbb{N}$, $\left. \begin{array}{l} \textcircled{1} 1 \in S \\ \textcircled{2} k \in S \Rightarrow k+1 \in S \end{array} \right\} \Rightarrow S = \mathbb{N}$

(Prop) WOP \Leftrightarrow PI

(pf) (\Rightarrow) Suppose $S \neq \mathbb{N}, \Rightarrow \mathbb{N} \setminus S \neq \emptyset$.

By WOP, $\mathbb{N} \setminus S$ has the least element $n \Rightarrow n-1 \notin \mathbb{N} \setminus S$
 $\Rightarrow n-1 \notin \mathbb{N}$ or $n-1 \in S$.

But $n \neq 1, \Rightarrow n-1 \in \mathbb{N} \Rightarrow n-1 \in S$

By PI, $\Rightarrow n \in S$ (\rightarrow)

(\Leftarrow) Let $S \subseteq \mathbb{N}$ have no least element

$$1 \notin S \Rightarrow \underline{1 \in \mathbb{N} \setminus S} \quad \text{--- ①}$$

Suppose $k \in S$, then $k+1 \notin S$ (or else $k+1$ is the least element of S)

$$\Rightarrow \underline{k+1 \in \mathbb{N} \setminus S} \quad \text{--- ②}$$

By ① + ②: $PI \Rightarrow \mathbb{N} \setminus S = \mathbb{N} \Rightarrow S = \emptyset$