Online Math Camp (2/25)

TA Session Notes (3/13)

$\inf E = \sup(-E)$

Cauchy-Schwarz Inequility

$\forall\, a, b \in \mathbb{C}^n,\quad |a|^2 \cdot |b|^2 \geq |\langle a, b\rangle|^2$

(pf) For $\mathbb{R}^2$: Need to show $(a_1^2 + a_2^2)(b_1^2 + b_2^2) \geq (a_1 b_1 + a_2 b_2)^2$

$$\Leftrightarrow a_1^2 b_1^2 + a_2^2 b_1^2 + a_1^2 b_2^2 + a_2^2 b_2^2 \geq a_1^2 b_1^2 + 2 a_1 b_1 a_2 b_2 + a_2^2 b_2^2$$

$$\Leftrightarrow a_2^2 b_1^2 + a_1^2 b_2^2 - 2 a_2 b_1 a_1 b_2 \geq 0$$

$$\Leftrightarrow (a_2 b_1 - a_1 b_2)^2 \geq 0 \quad \#$$

For $\mathbb{R}^n$: To show $(\sum a_i^2)(\sum b_i^2) \geq (\sum a_i b_i)^2$,

Consider $\sum (a_i - b_i x)^2 = (\sum b_i^2) x^2 - 2(\sum a_i b_i) x + (\sum a_i^2) \geq 0$

Let $x = \dfrac{\langle a, b\rangle}{\langle b, b\rangle}$, $\Rightarrow \langle b, b\rangle \cdot \dfrac{\langle a, b\rangle^2}{\langle b, b\rangle^2} - 2\langle a, b\rangle \cdot \dfrac{\langle a, b\rangle}{\langle b, b\rangle} + \langle a, a\rangle \geq 0$

$$\Rightarrow \langle a, a\rangle - \frac{\langle a, b\rangle^2}{\langle b, b\rangle} \geq 0 \quad \Leftrightarrow |a|^2 \cdot |b|^2 \geq |\langle a, b\rangle|^2 \quad \#$$

(pf) (Continued) For $\mathbb{C}^n$,

consider $\sum \langle (a_i - b_i x), (a_i - b_i x) \rangle = \sum (a_i - b_i x)\overline{(a_i - b_i x)}$

$$= \sum (a_i \overline{a_i} - \overline{a_i} b_i x - a_i \overline{b_i}\, \overline{x} + b_i \overline{b_i}\, x \overline{x})$$

$$= \sum [\langle a_i, a_i \rangle - \langle b_i, a_i \rangle x - \langle a_i, b_i \rangle \overline{x} + \langle b_i, b_i \rangle \langle x, x \rangle]$$

$$= \langle a, a \rangle - \langle b, a \rangle x - \langle a, b \rangle \overline{x} + \langle b, b \rangle \langle x, x \rangle \geq 0$$

Let $x = \dfrac{\langle a, b \rangle}{\langle b, b \rangle} \Rightarrow \langle a, a \rangle - \dfrac{|\langle a, b \rangle|^2}{\langle b, b \rangle} - \dfrac{|\langle a, b \rangle|^2}{\langle b, b \rangle} + \langle b, b \rangle \dfrac{|\langle a, b \rangle|^2}{\langle b, b \rangle^2}$

$$= \langle a, a \rangle - \frac{|\langle a, b \rangle|^2}{\langle b, b \rangle} \geq 0 \quad \#$$

Well Ordered Principle (WOP)

$\forall S \subseteq \mathbb{N},\ S \neq \phi, \Rightarrow S$ has the least element

Principle of Induction (PI)

If $S \subseteq \mathbb{N}$, $\left.\begin{array}{l} ① \ 1 \in S \\ ② \ k \in S \Rightarrow k+1 \in S \end{array}\right\} \Rightarrow S = \mathbb{N}$

(Prop) WOP $\Leftrightarrow$ PI

(pf) $(\Rightarrow)$ Suppose $S \neq \mathbb{N}$, $\Rightarrow \mathbb{N} \setminus S \neq \phi$.

By WOP, $\mathbb{N} \setminus S$ has the least element $n \Rightarrow n-1 \notin \mathbb{N} \setminus S$

$\Rightarrow n-1 \notin \mathbb{N}$ or $n-1 \in S$.

But $n \neq 1$, $\Rightarrow n-1 \in \mathbb{N} \Rightarrow n-1 \in S$

By PI, $\Rightarrow n \in S$ $(\rightarrow\leftarrow)$

$(\Leftarrow)$ Let $S \subseteq \mathbb{N}$ have no least element

$1 \notin S \Rightarrow \underline{1 \in \mathbb{N} \setminus S}$ —— ①

Suppose $k \notin S$, then $k+1 \notin S$ (or else $k+1$ is the least element of S)

$\Rightarrow \underline{k+1 \in \mathbb{N} \setminus S}$ —— ②

By ① + ②: PI $\Rightarrow \mathbb{N} \setminus S = \mathbb{N} \Rightarrow S = \emptyset$ #