Online Math Camp (235)

Quiz 3 Solution (3/13)

1. (30 pts) State and prove *the Cauchy-Schwarz inequality.*

See Review Notes of 3/13.

2. (30 pts) Let $z_1, z_2 \cdots, z_n$ be complex numbers, prove that

$$|z_1 + \cdots + z_n| \le |z_1| + \cdots + |z_n|$$

For $n=2$, need to show $|z_1+z_2| \le |z_1| + |z_2|$.

$$|z_1+z_2| = \sqrt{(z_1+z_2)\overline{(z_1+z_2)}} = \sqrt{z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2}$$

$$= \sqrt{|z_1|^2 + 2\mathrm{Re}(z_1\bar{z}_2) + |z_2|^2} \le \sqrt{|z_1|^2 + 2|z_1|\cdot|z_2| + |z_2|^2} = |z_1| + |z_2|$$

since $z_1\bar{z}_1 = |z_1|^2$ & $z_1\bar{z}_2 + \bar{z}_1 z_2 = 2\mathrm{Re}(z_1\bar{z}_2)$

For n, $|z_1 + (z_2 + \cdots z_n)| \le |z_1| + |z_2 + (z_3 + \cdots + z_n)| \le |z_1| + |z_2| + |z_3 + (z_4 + \cdots + z_n)|$

$$\le \cdots \le |z_1| + |z_2| + \cdots + |z_n| \quad \#$$

3. (27 pts) Prove the following statement, "Principle of Induction $\Rightarrow$ Well-Ordering Principle." (Hint. Use Induction and prove the base case as detailed as you can.)

(pf) PI: For $S \subseteq \mathbb{N}$ s.t. ① $1 \in S$
② If $k \in S$, then $k+1 \in S$ $\Bigg\} \Rightarrow S = \mathbb{N}$.

$\Downarrow$

WOP: Any non-empty set of $\mathbb{N}$ has a least element.

To show PI $\Rightarrow$ WOP:

Suppose $S \subseteq \mathbb{N}$ has no least element

① $1 \notin S$ (or that is the least element)

If $1 \notin S$, then $2 \notin S$ (or that is the least element), ...,

② If $1, 2, \dots, k \notin S$, then $k+1 \notin S$

Consider the set $\mathbb{N} \setminus S$, ①, ② $\Rightarrow$ ① $1 \in \mathbb{N} \setminus S$

② If $1 \sim k \in \mathbb{N} \setminus S$, then $k+1 \in \mathbb{N} \setminus S$

By PI, $\mathbb{N} \setminus S = \mathbb{N}$. i.e. $S = \phi$. #

4. (27 pts) Let $z = a + ib, w = u + iv$ and $z^2 = w$. Calculate a, b in terms of u, v. (Reminder. There are two roots.)

$$z^2 = (a+bi)^2 = \underline{(a^2-b^2)} + \underline{2ab}i = \underline{u} + \underline{v}i$$

$$\Rightarrow \begin{cases} a^2 - b^2 = u \\ 2ab = v \end{cases} \Rightarrow b^2 = \frac{-u+\sqrt{u^2+v^2}}{2}$$

If $v \geq 0$, $a = \pm\sqrt{\frac{u+\sqrt{u^2+v^2}}{2}}$, $b = \pm\sqrt{\frac{-u+\sqrt{u^2+v^2}}{2}}$

If $v < 0$, $a = \pm\sqrt{\frac{u+\sqrt{u^2+v^2}}{2}}$, $b = \mp\sqrt{\frac{-u+\sqrt{u^2+v^2}}{2}}$

5. (24 pts) Suppose z is a complex number with $|z| = 1$, calculate

$$|1+z|^2 + |1-z|^2,$$

and interpret it geometrically. (Hint. What is the geometric interpretation of $|a-b|$?)

Suppose $z = a+bi$. $|z| = 1 \Rightarrow a^2+b^2 = 1$

$$\Rightarrow |1+z|^2 + |1-z|^2 = |(a+1)+bi|^2 + |(1-a)-bi|^2 = [(a+1)+bi]\overline{[(a+1)+bi]} + [(1-a)-bi]\overline{[(1-a)-bi]}$$

$$= (a+1)^2 + b^2 + (1-a)^2 + b^2 = a+2a+1+b^2+1-2a+a^2+b^2$$

$$= 2(a^2+b^2)+2 = 4$$

Geometrically: