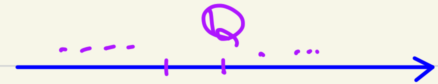
Online Math Camp (235)

TA Session Notes (3/6)

Construct $\mathbb{R}$

We "imagine" real numbers as a line 

We know $\sqrt{2} \notin \mathbb{Q}$, so there are gaps in $\mathbb{Q}$.

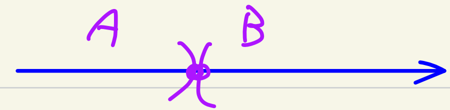
So the goal is to fill these gaps...

We look at 2 ways;

$\boxed{A}$ Dedekind Cut (Rudin way)

$\boxed{B}$ Sup (Least Upper Bound) (More convenient way)

$\boxed{A}$ Dedekind Cut



Intuition: If we cut a line into two parts, there must be a cutpoint.

Def: A cut (A, B) on set X is a pair of set such that:

① $A, B \neq \emptyset$, $A \cup B = X$ ($= \mathbb{Q}$, or $\mathbb{R}$ after it is constructed)

② $\exists\, a \in A$, $x \leq a \Rightarrow x \in A$. (A has upper bound a)

$\exists\, b \in B$, $x \geq b \Rightarrow x \in B$ (B has lower bound b)

Axiom of Dedekind Cut (on $\mathbb{R}$)

Exactly one the following holds: ① $\max A$ exists, ② $\min B$ exists.

Back to $\mathbb{Q}$ (Problem of $\mathbb{Q}$)

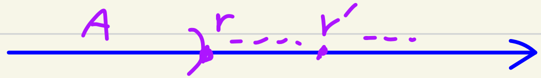
A cut on $\mathbb{Q}$: $A = \{x \in \mathbb{Q} \mid x^2 \leq 2,\ x > 0\} \cup \{x \in \mathbb{Q} \mid x \leq 0\}$

$B = \{x \in \mathbb{Q} \mid x^2 > 2\}$

Note: Neither $\underline{\max A}$, nor $\underline{\min B}$ exists in $\mathbb{Q}$ (gap!)

B Sup (Least Upper Bound)

Intuition:



Def: $r = \sup A$ if ① r is a upper bound. i.e. $\forall a \in A$, $r > a$.

② $\forall r' < r$, $\exists a \in A$ such that $r' < a$

Least Upper Bound Property A set has LUB property if

"$\forall S' \subseteq S$, $S' \neq \emptyset$, S' has a upper bound $\Rightarrow \sup S' \in S$ exists

Note: $\mathbb{R}$ has LUB property, but $\mathbb{Q}$ does not.

We now prove 4 propositions related to the LUB property

Proposition: $\mathbb{N}$ has no upper bound.

(pf) Suppose $\mathbb{N}$ has a upper bound.

By LUB property of $\mathbb{R}$, $\sup \mathbb{N} = r$ exists.

Hence, $\exists\, n \in \mathbb{N}$ such that $n > r-1$ (since $r-1 < r$ is not an upper bound)

$\Rightarrow n+1 > r$ $(\rightarrow\leftarrow)$

Proposition: (Archimedean Property) $\forall\, x, y > 0$, $\exists\, n \in \mathbb{N}$ such that $nx > y$.

(pf) Suppose the property fails for some $x, y > 0$

$\Rightarrow y$ is an upper bound of $\{nx \mid n \in \mathbb{N}\}$

$\Rightarrow \sup\{nx \mid n \in \mathbb{N}\} = r$ exists

$\Rightarrow \exists\, n \in \mathbb{N}$, such that $nx > r - x$

$\Rightarrow (n+1)x > r$ $(\rightarrow\leftarrow)$

Proposition: $\inf\{\frac{1}{n} \mid n\in\mathbb{N}\} = 0$

(pf) Suppose $\inf\{\frac{1}{n} \mid n\in\mathbb{N}\} = \varepsilon > 0 \Rightarrow \forall n\in\mathbb{N},\ \frac{1}{n} \geq \varepsilon.$

$\Rightarrow \forall n\in\mathbb{N}\quad n \leq \frac{1}{\varepsilon}.$ i.e. $\mathbb{N}$ has an upper bound. $(\rightarrow\leftarrow)$

Example $\{\frac{n}{n+1} \mid n\in\mathbb{N}\} = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots\}$ has $\sup = 1$, $\inf = \frac{1}{2}$.

Proposition: (Denseness of $\mathbb{Q}$) $\forall x, y\in\mathbb{R}$, $x<y \Rightarrow \exists q\in\mathbb{Q}$ such that $x<q<y$

(pf) (Intuition: Want to find $x<\frac{m}{n}<y \Leftrightarrow nx<m<ny \Leftrightarrow n(y-x)>1 \ldots$)

Pick $n\in\mathbb{N}$ such that $n(y-x)>1$.

Since $\exists\, m_1, m_2\in\mathbb{Z}$ such that $m_2<nx<m_1$, ($\mathbb{N}$ has no upper bound $\Rightarrow \mathbb{Z}$ has no lower or upper bound)

$\sup\{k\in\mathbb{N} \mid k\leq nx\} = m-1$ exists.

$$\left.\begin{aligned} &\Rightarrow m-1\leq nx<m. \\ &\Rightarrow ny>nx+1\geq(m-1)+1=m \end{aligned}\right\} \Rightarrow nx<m<ny \quad \#$$

Useful Techniques:

1. To prove "$x = y$", prove instead: "$x \geq y$" & "$y \geq x$"

2. To prove "$x \geq y$", prove instead: "$\forall n \in \mathbb{N}, \ x + \frac{1}{n} > y$" $\iff y - x < \frac{1}{n} \ \forall n \in \mathbb{N}$

$\iff y - x \leq 0$

$\iff y \leq x$