Online Math Camp (23S)

TA Session Notes (3/6)

[Quiz 2 Solution]

1. (30 pts, 15pts each) Give formal definitions to the following statements.

(i) a is the *least upper bound* of the set $S \subset \mathbb{R}$.

(ii) $S \subset \mathbb{R}$ satisfies *the least upper bound property.*

1.

(i) a is the least upper bound of S if

(1) a is an upper bound of S

(2) $\forall \gamma < a$, γ is not an upper bound of S.

(ii) S is said to satisfy the least upper bound property if every non-empty subset having upper bound has least upper bound.

2. (30 pts, 15pts each) Let $A = \left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{N} \right\}$. What are $\sup A$, $\inf A$?

$\sup A = 1$: $1 \geq \frac{1}{n}$ $\forall n \in \mathbb{N}$. Also, $1 \in A$.

Since for any $x < 1$, x is not an upper bound of A. $\Rightarrow \sup A = 1$.

$\inf A = 0$: 0 is a lower bound since $0 \leq \frac{1}{n}$ $\forall n \in \mathbb{N}$

3. (30 pts) For $E \subseteq \mathbb{R}$, prove that

$$\inf E = -\sup(-E).$$

(pf) $a = \inf E$. i.e. (1) a is a lower bound of E

(2) $\forall r > a$, r is not a lower bound of E

Need to show that ① $-a$ is a upper bound of E.

② $\forall r < -a$, r is not a upper bound of E.

For ①, (1) $\Rightarrow \forall x \in E$, $x \geq a$ $\Rightarrow \forall \underline{-x} \in -E$, $\underline{-x} \leq -a$. (definition)

$\Rightarrow \forall \underline{y} \in -E$, $\underline{y} \leq -a$. ($y = -x$) $\Rightarrow$ ① #

For ②, (2) $\Rightarrow \forall r > a$, $\exists x \in E$ such that $r > x$ (definition)

$\Rightarrow \forall \underline{-r} < -a$, $\exists \underline{-x} \in -E$ such that $\underline{-r} < \underline{-x}$ (add "$-$")

$\Rightarrow \forall \underline{\beta} < -a$, $\exists \underline{y} \in -E$ such that $\underline{\beta} < \underline{y}$ $\begin{pmatrix} \beta = -r \\ y = -x \end{pmatrix}$ $\Rightarrow$ ② #

4. Let $a > 1$. We assume that $a^{1/n}$ is already a well-defined notion in the following context for $n \in \mathbb{N}$, which denotes the unique positive solution of $x^n = a$.

(i) (14 pts) If m, n, p, q are integers, $n > 0$, $q > 0$, and $r = m/n = p/q$, prove that

$$(a^m)^{\frac{1}{n}} = (a^p)^{\frac{1}{q}}.$$

(pf) $\left[(a^m)^{\frac{1}{n}}\right]^{nq} = \left\{\left[(a^m)^{\cancel{\frac{1}{n}}}\right]^{\cancel{n}}\right\}^{q} = (a^m)^q = a^{mq}$

$\left[(a^p)^{\frac{1}{q}}\right]^{nq} = \left\{\left[(a^p)^{\cancel{\frac{1}{q}}}\right]^{\cancel{q}}\right\}^{n} = (a^p)^n = a^{pn} = a^{mq}$ by $\frac{m}{n} = \frac{p}{q}$

By the uniqueness of nq-th root, $(a^m)^{\frac{1}{n}} = (a^{mq})^{\frac{1}{nq}} = (a^p)^{\frac{1}{q}}$ #

(ii) (10 pts) Prove that $a^{r+s} = a^r a^s$ if r and s are rational.

(pf) Assume $r = \frac{m}{n}$, $s = \frac{p}{q}$.

$$\left(a^{r+s}\right)^{nq} = \left(a^{\frac{qm+pn}{nq}}\right)^{nq} = \left[\left(a^{\cancel{\frac{1}{nq}}}\right)^{\cancel{nq}}\right]^{qm+pn} = a^{qm+pn}$$

$$\left(a^r \cdot a^s\right)^{nq} = \left(a^{\frac{m}{n}}\right)^{nq} \cdot \left(a^{\frac{p}{q}}\right)^{nq} = \left\{\left[\left(a^m\right)^{\cancel{\frac{1}{n}}}\right]^{\cancel{n}}\right\}^q \cdot \left\{\left[\left(a^p\right)^{\cancel{\frac{1}{q}}}\right]^{\cancel{q}}\right\}^n$$

$$= a^{mq} \cdot a^{pn} = a^{mq+pn}.$$

By the uniqueness of nq-th root, we have $a^{r+s} = a^r \cdot a^s$. #

(iii) (14 pts) If x is real, define $A(x)$ to be the set of all numbers a^t, where t is rational and $t \leq x$. Prove that

$$a^r = \sup A(r)$$

when r is rational. Hence it makes sense to define

$$a^x = \sup A(x)$$

for every real x.

(pf) Need to prove $\forall\, t < r$, $t \in \mathbb{Q}$, $a^t < a^r$.

Let $t = \frac{m}{n} < \frac{p}{q} = r$ $(n, q > 0)$, $\Rightarrow mq < np$, $\Rightarrow a^{mq} < a^{np}$

Since $\underbrace{a^{np} - a^{mq}}_{>0} = \left(a^{\frac{p}{q}} - a^{\frac{m}{n}}\right) \underbrace{\left[\left(a^{\frac{p}{q}}\right)^{nq-1} + \left(a^{\frac{p}{q}}\right)^{nq-2} \cdot \left(a^{\frac{m}{n}}\right) + \cdots + \left(a^{\frac{m}{n}}\right)^{nq-1}\right]}_{>0}$

we have $a^{\frac{p}{q}} - a^{\frac{m}{n}} > 0$. $\Rightarrow a^r > a^t$.

Since $a^r \in A(r)$, $\sup A(r) = a^r$. #

(iv) (10 pts) Prove that $a^x a^y = a^{x+y}$ for all $x, y \in \mathbb{R}$.

(pf) $a^{x+y} = \sup A(x+y) = \sup\{a^t \mid t \in \mathbb{Q},\ t \leq x+y\}$ (by definition) ……(1)

$a^x \cdot a^y = \sup A(x) \cdot \sup A(y)$

$= \sup\{a^r \mid r \in \mathbb{Q},\ r \leq x\} \cdot \sup\{a^s \mid s \in \mathbb{Q},\ s \leq y\}$ (by definition)

$= \sup\{\underline{a^r \cdot a^s}_{= a^{r+s}} \mid r, s, \mathbb{Q},\ r \leq x,\ s \leq y\}$ ……(2)

Claim: $(1) = \sup\{a^t \mid t \leq x+y,\ t \in \mathbb{Q}\} = \sup\{a^t \mid t < x+y,\ t \in \mathbb{Q}\}$

$(2) = \sup\{a^{r+s} \mid r \leq x,\ s \leq y,\ r, s \in \mathbb{Q}\} = \sup\{a^{r+s} \mid r < x,\ s < y,\ r, s \in \mathbb{Q}\}$

Then, $\{t \mid t < x+y,\ t \in \mathbb{Q}\} = \{r+s \mid r < x,\ s < y,\ r, s \in \mathbb{Q}\}$,

$\Rightarrow \{a^t \mid t < x+y,\ t \in \mathbb{Q}\} = \{a^{r+s} \mid r < x,\ s < y,\ r, s \in \mathbb{Q}\}$,

So, $(1) = \sup\{a^t \mid t < x+y,\ t \in \mathbb{Q}\} = \sup\{a^{r+s} \mid r < x,\ s < y,\ r, s \in \mathbb{Q}\} = (2)$ #

Why is the Claim true? See video...

$$\underset{x+y}{\{t \mid t < x+y,\ t \in \mathbb{Q}\}} = \underset{x+y}{\{r+s \mid r < x,\ s < y,\ r, s \in \mathbb{Q}\}}$$

$$\{a^t \mid t < x+y,\ t \in \mathbb{Q}\} = \{a^{r+s} \mid r < x,\ s < y,\ r, s \in \mathbb{Q}\}.$$

$\Rightarrow \sup(1) = \sup(2)$

$\forall x < a^r$, x is not u.b. of (1). $\frac{1}{n} \longrightarrow 0$

$$a^r - a^t = a^t(a^{r-t} - 1) < \underbrace{a^r(a^{r-t} - 1)}_{\to 0},\ r > t.$$

By density prop. of $\mathbb{Q}$, $\exists\, t_n$ s.t. $r - \frac{1}{n} < t_n < r$

$$(a-1) = (a^{\frac{1}{n}} - 1)\underbrace{((a^{\frac{1}{n}})^{n-1} + (a^{\frac{1}{n}})^{n-2} + \cdots + 1)}_{> n}$$

$$\Rightarrow a^{\frac{1}{n}} - 1 < \frac{a-1}{n}$$

$$a^r - a^{t_n} < a^r(a^{r-t_n} - 1) < a^r(a^{\frac{1}{n}} - 1) < \frac{(a-1)a^r}{n} \searrow 0$$

$\exists n$ s.t. $a^r - a^{t_n} < a^r - x \Rightarrow a^{t_n} > x$

So x is not u.b. of (1), $\sup(1) = a^r$. #