Introduction to Real Analysis, Quiz 9

1. Define "X is a *complete metric space*".

   Solution. A metric space X is complete, if every Cauchy sequence converge to a point of X. ■

2. What are the lim sup and lim inf for the following sequences?

   (i) $a_n = \dfrac{(-1)^n}{1+\frac{1}{n}}$

   Solution. $\limsup\{a_n\} = 1$, $\liminf\{a_n\} = -1$. ■

   (ii) $a_n = \dfrac{1-2+3-4+\cdots+(-1)^{n-1}n}{n}$

   Solution. $\limsup\{a_n\} = 1/2$, $\liminf\{a_n\} = 1/2$. ■

   (iii) $a_n = \dfrac{n^2+4n-3}{2n^2+3n+5}$

   Solution. $\liminf = \limsup = 1/2$. ■

3. Discuss if the following series converge or diverge.

   (i) $\displaystyle\sum_{n=0}^{\infty} \frac{n}{2n+1}$.

   Solution. Diverge, since $\lim \dfrac{n}{2n+1} = \dfrac{1}{2} \neq 0$. ■

   (ii) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^{\frac{n}{2}}}$

   Solution. Converge, since $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^{\frac{n}{2}}} = \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^n = \frac{1}{1-\frac{1}{\sqrt{2}}}$. ■

   (iii) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}$

Solution. Converge, since $\sum_{n=0}^{\infty} \frac{1}{n!} < \sum_{n=0}^{\infty} \frac{1}{2^n} = 2$. In fact, the limit is e≈ 2.7182. ■

4. Say $|a_n| < 1$ for all $n \in \mathbb{N}$. Prove that the series $\sum a_n x^n$ converges for all x with $|x| < 1$.

   Solution. Using the comparison test, $|a_n| < 1$ for all $n \in \mathbb{N}$ implies that $|a_n x^n| \leq |x^n|$ for all $n \in \mathbb{N}$. And $\sum_{i=0}^{\infty} |x^n| = \sum_{i=0}^{\infty} |x|^n = \frac{1}{1-|x|}$(converges). Therefore, $\sum a_n x^n$ converges. ■

5. Calculate

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)(n+4)}$$

   Solution. First, observe that

$$\frac{1}{n(n+2)(n+4)} = \frac{1}{8}\left(\frac{1}{n} - 2\frac{1}{n+2} + \frac{1}{n+4}\right).$$

   Therefore,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n(n+2)(n+4)} &= \sum_{n=1}^{\infty} \frac{1}{8}\left(\frac{1}{n} - 2\frac{1}{n+2} + \frac{1}{n+4}\right) = \frac{1}{8}\left(\sum_{n=1}^{\infty} \frac{1}{n} - 2\sum_{n=1}^{\infty} \frac{1}{n+2} + \sum_{n=1}^{\infty} \frac{1}{n+4}\right) \\ &= \frac{1}{8}\left(\sum_{n=1}^{\infty} \frac{1}{n} - 2\sum_{n=1}^{\infty} \frac{1}{n} + \sum_{n=1}^{\infty} \frac{1}{n} + (2+1) - (1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4})\right) \\ &= \frac{1}{8}\left(3 - \frac{25}{12}\right) = \frac{11}{96}.\end{aligned}$$

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