1. Let $\{p_n\}$ be a sequence. What does it mean by $\lim_{n\to\infty} p_n = p$?

If $\exists p \in X$ s.t. $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $d(p_n, p) < \varepsilon$,

then we write $\lim_{n\to\infty} p_n = p$.

2. (20 pts each) Give a rigorous $N-\varepsilon$ argument to calculate the limits of the following sequences.

(i) $\sqrt{n+1}-\sqrt{n}$

(ii) $\dfrac{3n}{n+3}$

(i) $\lim_{n\to\infty}(\sqrt{n+1}-\sqrt{n})=0$: $|\sqrt{n+1}-\sqrt{n}|=\dfrac{1}{\sqrt{n+1}+\sqrt{n}}\leq\dfrac{1}{2\sqrt{n}}$.

$\forall\varepsilon>0$, take $n>\dfrac{1}{2\varepsilon^2}$, then $|\sqrt{n+1}-\sqrt{n}|<\varepsilon$.

(ii) $\lim_{n\to\infty}\dfrac{3n}{n+3}=3$: $\left|\dfrac{3n}{n+3}-3\right|=\dfrac{9}{n+3}$. $\forall\varepsilon>0$,

take $n>\max(1,\dfrac{9}{\varepsilon}-3)$, then $\left|\dfrac{3n}{n+3}-3\right|<\varepsilon$.

#1

3. Prove that a sequence can have at most one limit.

Suppose a sequence $\{p_n\}$ has two limits p_1, p_2. If $d(p_1,p_2) \neq 0$, take $0 < \varepsilon < \frac{d(p_1,p_2)}{2}$, suppose when $n > N_1$, $d(p_n,p_1) < \varepsilon$, when $n > N_2$, $d(p_n,p_2) < \varepsilon$. Thus when $n > \max(N_1,N_2)$,

$$d(p_1,p_2) \leq d(p_1,p_n) + d(p_2,p_n) < 2\varepsilon < d(p_1,p_2) \quad (\rightarrow\leftarrow).$$

Hence $\{p_n\}$ can have at most one limit.

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4. (24 pts) Prove that in any metric space X, every convergent sequence is a Cauchy sequence.

For any convergence sequence $p_n \to p$. $\forall \varepsilon > 0$, $\exists N$ s.t.
$d(p_n, p) < \frac{\varepsilon}{2}$ $\forall n > N$. Thus $\forall n, m > N$

$$d(p_m, p_n) \leq d(p_m, p) + d(p, p_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$$

Hence p_n is Cauchy.

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5. Given $a_1 > b_1 > 0$. Define $a_n = \dfrac{a_{n-1} + b_{n-1}}{2}$, $b_n = \dfrac{2a_{n-1}b_{n-1}}{a_{n-1} + b_{n-1}}$ for $n \geq 2$. Prove that $a_n > a_{n+1} > b_{n+1} > b_n$ and deduce that two sequences $\{a_n\}$ and $\{b_n\}$ converge and share the same limit.

When $n=2$, $a_2 = \frac{a_1+b_1}{2} < a_1$. Also, $b_2 = \frac{2a_1b_1}{a_1+b_1} = \frac{a_1b_1}{a_2}$

$\Leftrightarrow a_1b_1 = a_2b_2$. Since $a_1 > a_2 > 0$, $b_1 < b_2$. Last,

$a_2 > b_2 \Leftrightarrow (a_1+b_1)^2 > 4a_1b_1 \Leftrightarrow (a_1-b_1)^2 > 0$.

Suppose it is true when $n=k$, then when $n=k+1$:

By induction hypothesis, $a_k > b_k$, so $a_{k+1} = \frac{a_k+b_k}{2} < a_k$.

On the other hand, since $a_kb_k = a_{k+1}b_{k+1}$, $b_k < b_{k+1}$.

Finally, $a_{k+1} > b_{k+1} \Leftrightarrow (a_k-b_k)^2 > 0$. Hence by M.I,

$a_n > a_{n+1} > b_{n+1} > b_n \quad \forall n$.

Claim: $\lim\limits_{n\to\infty} b_n = \sup b_n$.

subpf: Since b_n is bounded and non-empty by least upper bound property, $\sup b_n = b$ exists, so $\forall \varepsilon > 0$, there exists N s.t. $b_N > b - \varepsilon$ $\forall n > N$. Also, $|b - b_n| \leq |b - b_N| < \varepsilon$. Hence $\lim\limits_{n\to\infty} b_n = b$.

Similarly, $\lim_{n\to\infty} a_n = \inf a_n$. Hence the limits of a_n and b_n exists. Suppose $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$. Thus

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{a_{n-1}+b_n}{2} \Rightarrow a = \frac{a+b}{2} \Rightarrow a = b.$$

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