Introduction to Real Analysis, Quiz 7

Lan sean

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1. Define "C is a *connected set* in the metric space X".

   Solution. C is connected if C is not an union of two non-empty separated sets. (Call A and B separated if $A \cap \overline{B} = \phi$ and $\overline{A} \cap B = \phi$.) ■

2. (a) State *Heine-Borel theorem.*

   Solution. In $\mathbb{R}^n$, K is compact if and only if K is closed and bounded. ■

   (b) Is $([a,b],d)$ compact where d denotes the discrete metric? Why you cannot use Heine-Borel in this case?

   Solution. Consider the open covering $\{N_1(x) : x \in [a,b]\}$ of $[a,b]$. Which has no finite subcover since $N_1(x) = \{x\}$.
   We cannot use Heine-Boral theorem in this case because it is not Euclidean matric space. ■

3. Prove that if a set is compact, then every infinite subset has a limit point.

   Solution. If not. Let S be the infinite subset of a compact set H and suppose S has no limit points. Since S has no limit point, $\forall p \in S$, $\exists N_{\epsilon_p}(p) \cap S = \{p\}$. Consider the open covering $\{H \setminus S, N_{\epsilon_p}(p), \forall p \in S\}$ of H, which has no finite subcover. Hence the statement is correct. ■

4. Show that the Cantor set is perfect, that is, closed and with no isolated point.

Solution. For closeness.
Let $F_1 = [0,1]$, $F_2 = [0,\frac{1}{3}] \cap [\frac{2}{3},1]$, $F_3 = [0,\frac{1}{9}] \cap [\frac{2}{9},\frac{1}{3}] \cap [\frac{2}{3},\frac{7}{9}] \cap [\frac{8}{9},1], \dots$
$F = \cup_{i=1}^{\infty} F_i$ is closed since F_i is closed for all i.

(no isolated point)
For $x \in F$, consider $(x-\epsilon, x+\epsilon)$. $\exists N \in \mathbb{N}$ such that $\frac{1}{3^N} < \epsilon$. Let $M = \max\{m : \frac{m}{3^N} < x, n \in \mathbb{N}\}$. We have $x \in [\frac{M}{3^N}, \frac{M+1}{3^N}] \subset (x-\epsilon, x+\epsilon)$. Now, removing $(\frac{3M+1}{3^N}, \frac{3M+2}{3^N})$ from $[\frac{M}{3^N}, \frac{M+1}{3^N}]$. If $x \in [\frac{M}{3^N}, \frac{3M+1}{3^N}]$, $\exists c \in F$ such that $c \in [\frac{3M+2}{3^N}, \frac{M+1}{3^N}]$. If $x \in [\frac{3M+2}{3^N}, \frac{M+1}{3^N}]$, $\exists c \in F$ such that $c \in [\frac{M}{3^N}, \frac{3M+1}{3^N}]$. Hence $(x-\epsilon, x+\epsilon) \cap F \setminus \{x\} \neq \phi$. ■

5. Prove that, if C is connected, then $\overline{C}$ is also connected. How about the inverse?

Solution. Let $\overline{C} = A \cup B$ and $\overline{A} \cap B = A \cap \overline{B} = \phi$. We want to show either $A = \phi$ or $B = \phi$. $C = (A \cap C) \cup (B \cap C)$ and note that $A \cap C$ and $B \cap C$ are separated. Since C is connected, WLOG, suppose $A \cap C = \phi$ and thus $C \in B \cap C$, which implies $C \subset B$ and $\overline{C} \subset \overline{B}$. $A = A \cap (A \cup B) = A \cap \overline{C} \subset A \cap \overline{B} = \phi$.

The inverse is not true. Consider $C = [-1,1] \setminus \{0\}$. $\overline{C} = [-1,1]$ is connected, but C is not. ■

Extra. State and prove *Heine-Borel theorem.*

Solution. (Statement): In $\mathbb{R}^n$, K is compact if and only if K is closed and bounded.
(Proof):

- If K is compact, we want to show K is closed and bounded. (Notably, this statement is always correct not only for $\mathbb{R}^n$)

  For K is bounded, consider an open cover $\{N_n(p)\}_{n=1}^{\infty}$ $(p \in K)$. Since K is compact, there is a finite subcover $\{N_{n_i}(p)\}_{i=1}^{N}$. Hence $K \subset N_{n_N}(p)$, K is bounded.

  For K is closed, we want to show that any point $p \notin K$ is not a limit point of K. For all point $q \in K$, let $U_q = N_{r/2}(q)$, $V_q = N_{r/2}(p)$, where $r = d(p,q)$. $\{U_q\}_q$ is open cover of K. Since K is compact, there is a finite subcover $\{U_{q_i}\}_{i=1}^{N}$. Let $V = \cap_{i=1}^{N} V_{q_i}$. If $s \in V$, then $s \in V_{q_i}$ for all i and $s \notin K \subset U_{q_i}$ for all i. Hence $V \cap K = \phi$, K is closed.

- If K is closed and bounded in $\mathbb{R}^n$.

  The proof structure is: if we can show $[a,b]$ is compact, then sine K is bounded, we can find $K \subset [a,b]$ for some a, b. Moreover, we know that any closed subset of a compact set is compact (if you don't know, you'll know later) Therefore, K is a closed subset of a compact set $[a,b]$, which implies that K is compact.

  First, show that $[a,b]$ is compact. Suppose not, let $\{G_\alpha\}$ be an open cover with no finite subcover. Then either $[a,c]$ or $[c,b]$ have no finite subcover. WLOG, let $[a,c]$ has no finite subcover. Repeat this method, we will get $I_1 = [a,b] \supset I_2 = [a,c] \supset \cdots$, and I_i has no finite subcover for all i. $\exists x^* \in I_n$ for all n (the existance of x^* can be show by proving $\sup \inf(I_i \,|\forall i) \in I_i$ for all i). Note that $x^* \in G_{\alpha_0}$ for some α_0. Since G_{α_0} is open, $\exists N_\epsilon(x^*) \subset G_{\alpha_0}$. $\exists I_m$ such that $I_m \subset N_\epsilon(x^*) \subset G_{\alpha_0}$. Contradict to G_α has no finite subcover to I_m.

  Second, show that a closed set of a compact set is compact. $B \subset K \subset X$, where B is closed and K is compact. Let $\{G_\alpha\}$ be a open cover of B, $\{G_\alpha\} \cap B^c$ is an open cover of K. Since K is compact, $\{G_{\alpha_i}\}_{i=1}^N \cap B^c$ finite subcover of K. And since $B \subset K$, $B \subset \{G_{\alpha_i}\}_{i=1}^N \cap B^c \implies B \subset \{G_{\alpha_i}\}_{i=1}^N$. Hence $\{G_{\alpha_i}\}$ is a finite subcover of B.

  Finally, complete the proof. Since K is bounded, $\exists [a_1, b_1] \times \cdots \times [a_n, b_n] \supset K$. Since K is closed by assumption and $[a_1, b_1] \times \cdots \times [a_n, b_n]$ is compact by the proof above, K is a closed subset of a compact set. Hence K is compact.

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