1. Define "K is a *compact set* in the metric space X".

A set in K is compact in X if every open cover of K has a finite subcover.

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2. Is the set S compact in X?

(i) $X = \mathbb{R}^2$. S is some open ball $N_r(x)$ for $r > 0$.

(ii) S is an empty set.

(iii) $X = \mathbb{R}^5$. S is a non-empty finite set.

(i) No. Consider an increasing sequence $\{r_n\}_{n\geq 1}$, $r_n = \frac{n}{n+1} r$, and $r_n \to r$. Then $\bigcup_{n\geq 1} N_{r_n}(x)$ is an open cover of $N_r(x)$ but does not have a finite subcover.

(ii) Yes.

(iii) Yes. Every finite set is compact.

3. Given X being a metric space and $K \subset Y \subset X$. Prove that K is compact in Y if and only if K is compact in X.

($\Rightarrow$): Suppose K is compact in Y. If $\{U_\alpha\}$ is an open covers of K in X, let $V_\alpha = U_\alpha \cap Y$, then $\{V_\alpha\}$ is an open cover of K in Y. Since K is compact in Y, $\exists$ a finite subcover $V_{\alpha_1} \cup \dots \cup V_{\alpha_n}$ in Y. $\Rightarrow U_{\alpha_1} \cup \dots \cup U_{\alpha_n}$ is a finite subcover of K in X.

($\Leftarrow$): Suppose K is compact in X. If $\{V_\alpha\}$ is an open cover of K in Y, since V_α is relative open in Y, $\exists U_\alpha$ open in X s.t. $V_\alpha = Y \cap U_\alpha$. We have $\{U_\alpha\}$ is an open cover of K in X, so $\exists$ a finite subcover $U_{\alpha_1} \cup \dots \cup U_{\alpha_n}$ of K in X $\Rightarrow V_{\alpha_1} \cup \dots \cup V_{\alpha_n}$ is a finite subcover of K in Y. #

4. Let F be a closed set and K be a compact set. Prove that $F \cap K$ is a compact set.

K is compact $\Rightarrow$ K is closed $\Rightarrow$ $F \cap K$ is a closed subset in compact set K $\Rightarrow$ $F \cap K$ is compact. #

5. Let $K = \{0\} \cup \left\{ \frac{1}{n} \middle| n \in \mathbb{N} \right\}$. Prove that K is compact (without using Heine-Borel if you know what it is).

Let $\{U_\alpha\}$ be any open cover of K, then $\exists\, G \in \{U_\alpha\}$ s.t. $0 \in G$. Hence $\exists$ sufficient large m s.t. $N_{\frac{1}{m}}(0) \subseteq G$.

Let G_n be a set s.t. $G_n \in \{U_\alpha\}$ and $\frac{1}{n} \in G_n$.

$\Rightarrow G, G_1, \ldots, G_{m-1}$ is a finite subcover of K

$\Rightarrow K$ is compact. #