Quiz 5 Answer key

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1. Give formal definitions to the following statements.

   (i) U is an *open set* in a metric space X.

   Solution. U is open in X if each point in U is a interior point of U. Equivalently, U is open in X if, for each point p in X, $\exists N(p)$ (a neighborhood of p) s.t. $N(p) \subset U$. ■

   (ii) F is a *closed set* in a metric space X.

   Solution. F is closed if every limit points of F is in F. ■

2. Is the set S open in X? Is it closed?

   (i) $X = \mathbb{R}^2$. S is some open ball $N_r(x)$ for $r > 0$.

   Solution. S is open.
   For each point $p \in N_r(x)$, consider $r_1 = r - d(p, x) > 0$. For each point $q \in N_{r_1}(p)$, $d(x, q) \leq d(x, p) + d(p, q) < d(x, p) + r_1 = r$. Thus $q \in N_r(x)$, which implies $N_{r_1}(p) \subset N_r(x)$. Hence $N_r(x)$ is open.
   S is **not** closed.
   For each point p in the circle of S, there is a point $p' \in C_{\epsilon/2} \subset N_\epsilon(p)$ such that $d(p', x) = \epsilon/2$. Hence the circle are limit points of S but not in S. ■

   (ii) S is X itself.

   Solution. S is open. For each point, its neighborhood is always in S since $S = X$.
   S is closed since every limit points in S is in X and $X = S$. ■

(iii) S is an empty set.

Solution. S is both open and closed since there is no point in S, the definitions are satisfied subsequently. ■

(iv) $X = \mathbb{R}^5$. S is a non-empty finite set.

Solution. S is **not** open.
Suppose S is open, then there exists $N_{r_1}(p) \subset S$, where $p \in S$. Let $r_i = 2^{-i+1} r_1$ and take $p_i \neq p$ such that $p_i \in N_{r_i}(p) \setminus N_{r_{i+1}}(p)$. Thus, we will have infinite points in S, contradict to S is finite.
S is closed.
Let $S = \{p_1, \ldots, p_n\}$ and define $r = \min |p_i - p_j : i \neq j|$. For each point p_i and $N_r(p_i)$, there is no another point $p_j \neq p_i$ such that $p_j \in N_r(p_i)$. Hence there is no limit point in S. ■

3. Prove that U is open if and only if U^c is closed.

Solution. ($\Longrightarrow$). If U is open. Suppose that U^c is not closed, that is, there is a limit point p of U^c which is in U. Since U is open and $p \in U$, $\exists N(p)$ s.t. $N(p) \subset U$, which is a contradiction to p is a limit point of U^c.

($\Longleftarrow$). If U^c is closed. For point $x \in U$, x is not a limit point of U^c. $\exists N(x)$ such that $N(x) \cap U^c \setminus \{x\} = \phi$, which implies $N(x) \subset U$. Hence U is open. ■

4. Show that the union of any collection of open sets is open.

Solution. Let $A = \cup_{i \in I} A_i$ where A_i is open set for all i. We want to prove that A is open set.

For any $a \in A$ there must exists at least one t such that $a \in A_t$. Since A_t is open, $\exists N(a)$ s.t. $N(a) \in A_t \in A$. The proof is complete. ■

5. Prove that a bounded closed set of real numbers contains its supremum and infimum.

Solution. Let S be a closed set and $\sup(A) = a$. Suppose $a \notin A$ (If not, then done.). Now, try to show that a is a limit point of A.
Since $a = \sup(A)$, for all $\epsilon > 0$, there is a $a' \in (a - \epsilon, a)$ s.t. $a' \in A$. (If not, then $a - \epsilon < a$ and $a - \epsilon$ is a upper bound of A, contradiction to $a = \sup(A)$.) Since ϵ is arbitrary, a is a limit point of A. Thus, $a \in A$ since A is closed.

The infimum case is similar. Let $b = \inf(A)$, where A is closed. Since $b = \inf(A)$, for all $\epsilon > 0$, there is a $b' \in (b, b + \epsilon)$ s.t. $b' \in A$. (If not, then $b + \epsilon > b$ and $b + \epsilon$ is a lower bound of A, contradiction to $b = \inf(A)$.) Since ϵ is arbitrary, b is a limit point of A. Thus, $b \in A$ since A is closed. ■